FIRST PROBLEM SET (due Friday, April 7th)

1. Show that the equation $x^2 + 2y^2 = 1$ has infinitely many solutions where $x, y \in \mathbb{Q}$. Give a parametric description of the set of solutions.

2. By using the “secant line method,” we have given a description of the set of solutions to the equation $x^2 - 2y^2 = 1$ in terms of a parameter $m$. Describe the values of $m$ which give a solution satisfying $x > 0$ and $y > 0$. Answer the same question for solutions satisfying $x < 0$ and $y > 0$.

3. Assume that $x, y \in \mathbb{Z}$ satisfy the equation $x^2 - 2y^2 = 1$ and that $x > 0, y > 0$. Prove the following inequalities.

$$\frac{1}{(2\sqrt{2} + \frac{1}{2})y^2} < |\sqrt{2} - \frac{x}{y}| < \frac{1}{2\sqrt{2} y^2}$$

4. Find a rational number $r$ such that $|\sqrt{2} - r| < 10^{-6}$ without using a calculator.

5. Let $s$ be a fixed rational number. Prove that there is a constant $c > 0$, depending only on $s$, such that

$$|s - \frac{x}{y}| > \frac{1}{cy}$$

for all rational numbers $\frac{x}{y} \neq s$. (Here $x, y \in \mathbb{Z}$ and $y > 0$.)

6. Let $s$ be a fixed rational number. Prove that there are only finitely many rational numbers $\frac{x}{y}$ (in reduced form) such that

$$|s - \frac{x}{y}| < \frac{1}{y^2}$$

7. The infinite series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to some real number $\alpha$. Prove that $\alpha$ is an irrational number.

8. Compute $N(\alpha)$ for the following elements $\alpha \in \mathbb{Q}[\sqrt{2}]$:

$$\frac{1}{2}, \sqrt{2}, \frac{1}{3} - \frac{2}{5\sqrt{2}}, 1 + \sqrt{2}, 5 + 3\sqrt{2}, 1 + 2\sqrt{2}$$

9. Assume that $n \in \mathbb{Z}$ and that there exists an element $\alpha \in \mathbb{Z}[\sqrt{2}]$ such that $N(\alpha) = n$. Prove that there exists an element $\beta \in \mathbb{Z}[\sqrt{2}]$ such that $N(\beta) = -n$. Prove that there exists an element $\gamma \in \mathbb{Z}[\sqrt{2}]$ such that $N(\gamma) = 2n$. 

1