SOLUTIONS FOR THE FIRST EXAMINATION IN MATH 480

QUESTION 1. Find the continued fraction expansion for $\sqrt{50}$.

SOLUTION: We write $\alpha_0 = \sqrt{50}$. Then $a_0 = [\alpha_0] = 7$. We obtain

$$\alpha_1 = \frac{1}{\alpha_0 - a_0} = \frac{1}{\sqrt{50} - 7} = \frac{\sqrt{50} + 7}{(\sqrt{50} - 7)(\sqrt{50} + 7)} = \frac{\sqrt{50} + 7}{1} = \sqrt{50} + 7$$

Hence $\alpha_2 = \sqrt{50} + 7$ and $a_1 = [\alpha_1] = [\sqrt{50} + 7] = 14$. Note that $\alpha_1 - a_1 = \sqrt{50} - 7$.

We obtain

$$\alpha_2 = \frac{1}{\alpha_1 - a_1} = \frac{1}{\sqrt{50} - 7} = \sqrt{50} + 7$$

just as before. Hence $\alpha_2 = \alpha_1$. Therefore $a_2 = a_1$ and the continued fraction algorithm will give $\alpha_n = \alpha_1$ and $a_n = a_1 = 14$ for all $n \geq 1$. That is, the continued fraction expansion for $\sqrt{50}$ is

$$\{7; 14, 14, 14, \ldots\}$$

QUESTION 2. This question concerns $\mathbb{Q}[\sqrt{5}]$ and $\mathbb{Z}[\sqrt{5}]$.

(a) Let $\beta = 2 + \sqrt{5}$. Suppose that $n$ is a positive integer. Since $\beta^n \in \mathbb{Z}[\sqrt{5}]$, we can write $\beta^n = a_n + b_n \sqrt{5}$, where $a_n, b_n \in \mathbb{Z}$. Note that $a_n, b_n > 0$. Prove that

$$a_n^2 - 5b_n^2 = (-1)^n$$

SOLUTION: For any $n \geq 1$, we have

$$a_n^2 - 5b_n^2 = N(a_n + b_n \sqrt{5}) = N(\beta^n) = N(\beta)^n$$

Note that $N(\beta) = N(2 + \sqrt{5}) = 2^2 - 1^25 = -1$. Hence we obtain

$$a_n^2 - 5b_n^2 = N(\beta)^n = (-1)^n$$

for any $n \geq 1$, just as we wanted to prove.
(b) With the same notation as in part (a), prove that if \( n \) is odd, then

\[
0 < \sqrt{5} - \frac{a_n}{b_n} < \frac{1}{b_n^2}
\]

SOLUTION: If \( n \) is odd, then \((-1)^n = -1\). Thus, by part (a), we have

\[
(a_n - b_n\sqrt{5})(a_n + b_n\sqrt{5}) = -1
\]

Divide this equation by \( b_n^2 \). We obtain

\[
\left(\frac{a_n}{b_n} - \sqrt{5}\right)\left(\frac{a_n}{b_n} + \sqrt{5}\right) = -\frac{1}{b_n^2}
\]

which is negative. Since \( a_n, b_n > 0 \), the second factor \( \left(\frac{a_n}{b_n} + \sqrt{5}\right) \) is clearly positive. Hence the first factor \( \left(\frac{a_n}{b_n} - \sqrt{5}\right) \) must be negative. That fact implies that

\[
\sqrt{5} - \frac{a_n}{b_n} > 0
\]

which is one of the inequalities to be proved. For the other inequality, note that \( \frac{a_n}{b_n} + \sqrt{5} > 1 \).

Therefore,

\[
\frac{1}{b_n^2} = \left| \left(\frac{a_n}{b_n} - \sqrt{5}\right)\right| \left| \left(\frac{a_n}{b_n} + \sqrt{5}\right)\right| > \left| \left(\frac{a_n}{b_n} - \sqrt{5}\right)\right|
\]

Thus, the positive number \( \sqrt{5} - \frac{a_n}{b_n} \) is strictly less than \( \frac{1}{b_n^2} \), which gives the second inequality to be proved.

(c) Suppose that \( m \in \mathbb{Q} \). Let \( \alpha = m + \sqrt{5} \in \mathbb{Q}[\sqrt{5}] \). Let \( \gamma = \alpha / \overline{\alpha} \). Since \( \alpha \neq 0 \), we can say that \( \gamma \in \mathbb{Q}[\sqrt{5}] \). Answer the following questions in the given order.

Show that \( N(\overline{\alpha}) = N(\alpha) \). Then show that \( N(\gamma) = 1 \).

Write \( \gamma = r + s\sqrt{5} \), where \( r, s \in \mathbb{Q} \). Determine \( r \) and \( s \) (in terms of \( m \)).

Show that \( r^2 - 5s^2 = 1 \). (There is a quick way and there is a long way. Your choice.)

SOLUTION: For any \( \alpha \in \mathbb{Q}[\sqrt{5}] \), we know that \( \overline{\alpha} = \alpha \). Therefore,

\[
N(\overline{\alpha}) = \overline{\alpha} \cdot \alpha = \alpha \cdot \overline{\alpha} = N(\alpha)
\]
Since $\gamma \cdot \overline{\alpha} = \alpha$, we can take norms and obtain $N(\gamma)N(\overline{\alpha}) = N(\alpha)$. Now $N(\overline{\alpha}) = N(\alpha)$ and $N(\alpha) \neq 0$ since $\alpha = m + \sqrt{5} \neq 0$ for any $m \in \mathbb{Q}$. It follows that $N(\gamma) = 1$.

We can divide since $\alpha \neq 0$. We obtain

$$\gamma = \frac{\alpha}{\overline{\alpha}} = \frac{\alpha^2}{\overline{\alpha} \alpha}$$

Now $\alpha^2 = (m + \sqrt{5})^2 = m^2 + 5 + 2m\sqrt{5}$ and $N(\alpha) = m^2 - 5$. Hence

$$\gamma = \frac{m^2 + 5 + 2m\sqrt{5}}{m^2 - 5} = \frac{m^2 + 5 + 2m}{m^2 - 5\sqrt{5}}$$

Therefore, $r = \frac{m^2 + 5}{m^2 - 5}$, $s = \frac{2m}{m^2 - 5}$.

Finally, note that $N(\gamma) = r^2 - 5s^2$. Also, as verified above, $N(\gamma) = 1$. Therefore, $r^2 - 5s^2 = 1$ as stated.

QUESTION 3. On the handout about continued fractions, one finds a definition of the function $f_n(x)$ for $n \geq 0$ associated to a sequence $a_0, a_1, \ldots$. We have discussed the functions $f_n(x)$ in class and proved several properties that they satisfy. One should use some of these properties in the following question. Please do not actually compute the numbers listed.

Arrange the following four rational numbers in increasing order.

$$s = \{1; 2, 3, 4, 5, 6\}, \quad t = \{1; 2, 3, 4, 5, 7\}, \quad u = \{1; 2, 3, 4, 5, 6, 7\}, \quad v = \{1; 2, 3, 4, 5, 6, 7, 8\}$$

SOLUTION: Let $f_4(x) = \{1; 2, 3, 4, 5, x\}$. Then, as a special case of a general theorem proved in class, we know that $f_4(x)$ is a strictly decreasing function of $x$ for $0 < x < \infty$. We have

$$s = f_4(6), \quad t = f_4(7), \quad u = f_4(6 + \frac{1}{7})$$

Since $6 < 6 + \frac{1}{7} < 7$, it follows that $s > u > t$. It remains to decide where to place $v$ in the ordering. We will mention two methods.

Notice that $v = f_4(\{6; 7, 8\})$ and $\{6; 7, 8\} = 6 + \frac{1}{7 + \frac{1}{8}} = 6 + \frac{1}{\frac{57}{8}} = 6 + \frac{8}{57}$. Also, notice that $6 < 6 + \frac{8}{57} < 6 + \frac{8}{56} = 6 + \frac{1}{7}$. Using the fact that $f_4(x)$ is strictly decreasing again, we obtain that $s > v > u$. Hence, we get the ordering

$$t < u < v < s$$
Alternatively, we can use the function $f_6(x) = \{1; 2, 3, 4, 5, 6, 7, x\}$ which is also a strictly decreasing function for $0 < x < \infty$. As discussed in class, the range of this function is the interval $\{1; 2, 3, 4, 5, 6, 7\} < y < \{1; 2, 3, 4, 5, 6\}$. That is, the range of $f_6(x)$ is the interval $u < y < s$. Since $v = f_6(8)$ is in the range of $f_6(x)$, we again obtain the inequalities $u < v < s$.

Thus, the four numbers arranged in increasing order are: $t, u, v, s$. 

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