Topics in Algebraic Number Theory

The main theme of this course will be the interplay between algebraic number theory and analytic number theory. One of the objects of study in algebraic number theory is the structure of the ring of integers of a number field \( F \). This includes the study of prime ideals and their residue fields, the order and structure of the ideal class group, and the structure of the group of units. Some of the information about these objects is encoded in the Dedekind \( \zeta \)eta function of \( F \). This will be one of our main objects of study. Here are some of the specific topics we will discuss:

Dirichlet \( L \)-functions. These functions were introduced by Dirichlet for the purpose of proving his famous theorem about primes in arithmetic progressions. A Dirichlet \( L \)-function is associated to every character of the finite abelian group \( U_m \), the group of invertible elements in \( \mathbb{Z}/m\mathbb{Z} \), where \( m \geq 1 \). The Riemann \( \zeta \)eta function is a special case. The character theory of finite abelian groups will be discussed. We will then prove certain results about the analytic continuation of Dirichlet \( L \)-functions. The proof of Dirichlet’s Theorem will be reduced to proving the nonvanishing of \( L(s, \chi) \) at \( s = 1 \) for all non-trivial Dirichlet characters \( \chi \).

The Dedekind \( \zeta \)eta function for a quadratic or cyclotomic extension of \( \mathbb{Q} \). One fundamental result is the relationship with Dirichlet \( L \)-functions. This result leads us to a proof of the non-vanishing of \( L(1, \chi) \) for non-trivial Dirichlet characters \( \chi \), and hence to the completion of the proof of Dirichlet’s Theorem. We will also eventually obtain a formula for the class number of an imaginary quadratic field.

A formula for the residue of the Dedekind \( \zeta \)eta function of a number field at \( s = 1 \). This is a very general formula. We will prove it for quadratic extensions of \( \mathbb{Q} \).

Artin \( L \)-functions. These functions can be viewed as a generalization of Dirichlet \( L \)-functions. They are associated with a finite Galois extension \( K/F \) of number fields and a finite-dimensional representation of the Galois group \( G = Gal(K/F) \). We will introduce what we need about the representation theory for finite groups in order to discuss this topic.

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OFFICE HOURS: Monday 12:00 - 1:00

GRADING: There will be 10 to 15 homework problems during the quarter. The course grade will depend on how many problems are completed correctly. The grade will be in the range 3.0 to 4.0.