The maximal ideals in the ring $\mathcal{F}_{cont}(\ [0,\ 1],\ \mathbf{R}\ )$.

Suppose that R is the ring of continuous real-valued functions on the interval $[0,\ 1]$. If $u \in [0,\ 1]$, then we can define

$$M_u \ = \ \{ f \in R \mid f(u) = 0 \ \} \ .$$

As explained in class one day, M_u is a maximal ideal in the ring R. Furthermore, if $u_1,\ u_2$ are in $[0,\ 1]$ and $u_1 \neq u_2$, then $M_{u_1} \neq M_{u_2}$. The purpose of this hand-out is to prove the following proposition.

Proposition: *Every maximal ideal of R is of the form M_u for some $u \in [0,\ 1]$.*

Proof: Suppose that M is a maximal ideal of R. We will prove that $M \subseteq M_u$ for some $u \in [0,\ 1]$. Since M is maximal and $M_u \neq R$, it will then follow that $M = M_u$.

Assume to the contrary that, for all $u \in [0,\ 1]$, $M \not\subseteq M_u$. That assumption means that, for every $u \in [0,\ 1]$, there exists at least one function $f \in M$ such that $f \notin M_u$. For each $u \in [0,\ 1]$, pick one such f and denote it by f_u. Thus, we have $f_u \in M$ and $f_u(u) \neq 0$.

Suppose $u \in (0,\ 1)$. Let $c = |f_u(u)|$. Thus, $c \in \mathbf{R}$ and $c > 0$. Since f_u is continuous at u, it follows that there exists an open interval $(a_u,\ b_u) \subseteq (0,\ 1)$ such that $u \in (a_u, b_u)$ and such that $|f_u(x)| > c/2$ for all $x \in (a_u,\ b_u)$. In particular, it follows that $f_u(x) \neq 0$ for all $x \in (a_u,\ b_u)$.

We must also consider f_0 and f_1, corresponding to the endpoints $u = 0$ and $u = 1$. Again, by using continuity, we see that there exist intervals $[0,\ b_0)$ and $(a_1,\ 1]$ such that $f_0(x) \neq 0$ for all $x \in [0,\ b_0)$ and $f_1(x) \neq 0$ for all $x \in (a_0,\ 1]$. We will enlarge those intervals by just choosing $a_0 < 0$ and $b_1 > 1$ arbitrarily.

With the above notation, we can say the following. If $u \in [0,\ 1]$, then $u \in (a_u,\ b_u)$ and there is an element $f_u \in M$ such that f_u is nonvanishing on $(a_u,\ b_u) \cap [0, 1]$. (Note that we need to take the intersection only when $u = 0$ or $u = 1$. For $0 < u < 1$, we have chosen the interval $(a_u,\ b_u)$ to be contained in $(0,\ 1)$.)

Every point $v \in [0,\ 1]$ is contained in at least one of the intervals $(a_u,\ b_u)$. Namely, v is certainly contained in the interval $(a_v,\ b_v)$. Therefore, we have

$$[0,\ 1] \ \subseteq \ \bigcup_{u \in [0,\ 1]} (a_u,\ b_u) \ .$$

Therefore, the collection of open intervals $(a_u,\ b_u)$ forms an *"open covering"* of the closed interval $[0,\ 1]$. There is a theorem in analysis (or topology) which asserts that we can choose finitely many of the above open intervals and still have a covering of the closed interval $[0,\ 1]$. (In the terminology of topology, this means that $[0,\ 1]$ is a *'compact set."*) That is, there exists a positive integer n and points $u_1, ..., u_n \in [0,\ 1]$ such that

$$[0,\ 1] \ \subseteq\ \bigcup_{j=1}^{n} (a_{u_j},\ b_{u_j})\ .$$

Now consider the function $g = \sum_{j=1}^{n} f_{u_j}^2$. Thus, we have

$$g(x) \ = \ \sum_{j=1}^{n} f_{u_j}(x)^2$$

for all $x \in [0,\ 1]$. First of all, note that g is a continuous function on $[0,\ 1]$ since each term in the above finite sum is continuous on $[0,\ 1]$. That is, $g \in R$. In fact, since each $f_{u_j} \in M$, so is its square. Hence it is clear that $g \in M$.

If $x \in [0, 1]$, then $x \in (a_{u_k},\ b_{u_k})$ for at least one $k,\ 1 \le k \le n$. Choose such a k. The function f_{u_k} is nonvanishing on the entire interval $(a_{u_k},\ b_{u_k}) \cap [0,\ 1]$. In particular, $f_{u_k}(x) \neq 0$. Thus, at least one of the terms is the sum defining $g(x)$ is positive. The other terms are obviously nonnegative. Hence it is clear that $g(x) > 0$.

We have proved that $g \in M$ and that $g(x) \neq 0$ for all $x \in [0,\ 1]$. Since g is continuous on the interval $[0,\ 1]$ and is nonvanishing on that entire interval, it follows from calculus that the function h on $[0,\ 1]$ defined by

$$h(x) \ = \ 1/g(x)$$

for all $x \in [0,\ 1]$ is also a continuous function on the interval $[0,\ 1]$. That is, $h \in R$. We have $gh = 1_R$, the constant function 1, and so it follows that g is an invertible element of R. However, this is impossible since $g \in M$ and $M \neq R$. This is a contradiction. Thus, a maximal ideal M which is not of the form M_u for some $u \in [0, 1]$ cannot exist. QED

The maximal ideals in the ring $\mathcal{F}_{cont}\big(\ (0,\ 1),\ \mathbf{R}\ \big)$.

The above proposition fails to be true for the ring $R = \mathcal{F}_{cont}\big(\ (0,\ 1),\ \mathbf{R}\ \big)$. The proof breaks down because the open interval $(0,\ 1)$ is not compact. As an example, the open intervals $(\frac{1}{n},\ 1)$, where n varies over all integers ≥ 2, form a covering of $(0,\ 1)$, but it is not possible to choose finitely many of those intervals and still have a covering of $(0,\ 1)$.

It is true that, for each $u \in (0,\ 1)$, the set M_u (consisting of the elements of R which vanish at u) is a maximal ideal of R. However, there do exist maximal ideals M in that ring which are not of the form M_u for any $u \in (0,\ 1)$. This is proved as follows. Let

$$I \ = \ \{f \in R \mid f(\tfrac{1}{n}) = 0 \ for\ all\ but\ finitely\ many\ n \in \mathbf{N}\ \}\ .$$

Note that $I \neq R$ since the constant function 1_R is not in I. Also, consider the function g defined by $g(x) = sin(\pi/x)$ for all $x \in (0,\ 1)$. That function vanishes at all the points $x = \frac{1}{n}$ for $n \in \mathbf{N}$. Thus, $g \in I$. Also, for any $m \in \mathbf{N}$, let h_m be defined by

$$h_m(x) \ = \ \begin{cases} sin(\pi/x) & for\ x \leq \frac{1}{m} \\ \quad . & \quad . \\ x - \frac{1}{m} & for\ x > \frac{1}{m} \end{cases}$$

Then $h_m \in R$, $\ h_m(\frac{1}{n}) = 0$ for all $n \in \mathbf{N}$ such that $n \geq m$, but $h_m(\frac{1}{n}) \neq 0$ for $n \in \mathbf{N}$ such that $n < m$. Also, $h_m(x) \neq 0$ for all $x \in (0,\ 1)$ not of the form $x = \frac{1}{n}$ for $n \in \mathbf{N}$.

It follows that $h_m(x) \in I$ for all $m \in \mathbf{N}$. However, for any $u \in (0,\ 1)$, it follows that $h_m(x) \notin M_u$ if m is chosen sufficiently large. Therefore, for any $u \in (0,\ 1)$, it follows that $I \not\subseteq M_u$. However, according to theorem 6 on the handout "Important theorems ...," there does exist at least one maximal ideal M of R containing I. Such a maximal ideal M cannot be of the form M_u for any $u \in (0,\ 1)$.

We haven't proved theorem 6 in class. The proof uses something called "Zorn's Lemma," a result from set theory which is equivalent to the Axiom of Choice. I will state Zorn's Lemma below. (It isn't the most general form.) We will need some terminology to state it.

Suppose that S is a nonempty set and $\mathcal{D}$ is a nonempty collection of subsets of S. Suppose that $\mathcal{C}$ is a nonempty subset of $\mathcal{D}$ with the property that if $U, V \in \mathcal{C}$, then either $U \subseteq V$ or $V \subseteq U$. We will then call $\mathcal{C}$ a *"chain in* $\mathcal{D}$*"* . Suppose that $M \in \mathcal{D}$. We say that M is a *"maximal element of* $\mathcal{D}$*"* if M has the following property: If $U \in \mathcal{D}$ and $M \subseteq U$, then $U = M$.

Zorn's Lemma: *Assume that, for every chain* $\mathcal{C}$ *in* $\mathcal{D}$*, the set* $\bigcup_{U \in \mathcal{C}} U$ *is in* $\mathcal{D}$*. Then* $\mathcal{D}$ *has at least one maximal element.*

To prove theorem 6, let I be any ideal of R, where R is a ring with unit. Assume that $I \neq R$. This means that $1_R \notin I$. We consider the following collection of subsets of R:

$$\mathcal{D} \;=\; \{\; J \mid J \textit{ is an ideal of } R \textit{ such that } \; I \subseteq J \;\; \textit{and} \;\; 1_R \notin J \;\}$$

It is clear that $\mathcal{D}$ is nonempty because I is an element of $\mathcal{D}$. It will be left as an exercise to prove that for every chain $\mathcal{C}$ in $\mathcal{D}$, the set $\bigcup_{J \in \mathcal{C}} J$ is an ideal of R containing I, but not containing 1_R. Therefore, according to Zorn's Lemma, $\mathcal{D}$ has at least one maximal element. Such a maximal element will be an ideal J of R containing I which has the following property: if J' is an ideal of R such that $J \subseteq J'$ and $1_R \notin J'$, then $J' = J$. That implies that J will be a maximal ideal of R containing I. Hence, indeed, there exist a maximal ideal of R containing I. This proves theorem 6.

As a final remark concerning the above example, apart from the maximal ideals M_u, it does not seem possible to describe the other maximal ideals M in any concrete or explicit way. Also, R/M will be a field, of course, but I don't know what field it will be.