SOLUTIONS FOR PROBLEM SET 3

**Section 6.4, Problem 5.** $b,\ c,\ f,$ (just the left cosets for part f),

**Solution.**

**(b)** There are two left cosets:

$$\{1+8\mathbb{Z},\quad 3+8\mathbb{Z}\},\qquad\qquad \{5+8\mathbb{Z},\quad 7+8\mathbb{Z}\}\quad .$$

These sets are also the right cosets.

**(c)** There are three left cosets:

$$3\mathbb{Z},\qquad\qquad 1+3\mathbb{Z},\qquad\qquad 2+3\mathbb{Z}\quad .$$

These sets are also the right cosets.

**(f)** There are three left cosets of $D_4$ in $S_4$:

$$D_4 \;=\; \{\; i,\;\; (1\ 2\ 3\ 4),\;\; (1\ 3)(2\ 4),\;\; (1\ 4\ 3\ 2),\;\; (1\ 2)(3\ 4),\;\; (1\ 4)(2\ 3),\;\; (1\ 3),\;\; (2\ 4)\;\},$$

$$(1\ 2)D_4 \;=\; \{\; (1\ 2),\;\; (2\ 3\ 4),\;\; (2\ 4\ 1\ 3),\;\; (1\ 4\ 3),\;\; (3\ 4),\;\; (1\ 4\ 2\ 3),\;\; (1\ 3\ 2),\;\; (1\ 2\ 4)\;\},$$

$$(1\ 4)D_4 \;=\; \{\; (1\ 4),\;\; (1\ 2\ 3),\;\; (1\ 3\ 4\ 2),\;\; (2\ 4\ 3),\;\; (1\ 2\ 4\ 3),\;\; (2\ 3),\;\; (1\ 3\ 4),\;\; (1\ 4\ 2)\;\}\quad .$$

**Section 6.4, Problem 6.**

**Solution.** We will use various properties of the determinant of a matrix. A left coset of $SL_2(\mathbb{R})$ in $GL_2(\mathbb{R})$ has the form

$$ASL_2(\mathbb{R}) \;=\; \{AB \;\big|\; B \in SL_2(\mathbb{R})\;\}$$

where $A \in GL_2(\mathbb{R})$. Let $d = det(A)$. Then $d \in \mathbb{R}$ and $d \neq 0$. If $B \in SL_2(\mathbb{R})$, then $det(B) = 1$. We then have $det(AB) = det(A)det(B) = d \cdot 1 = d$. Thus, all the elements in the above left coset have their determinant equal to $d$.

Conversely, suppose that $C \in GL_2(\mathbb{R})$ and $\det(C) = d$. Let $B = A^{-1}C$. Then $C = AB$. Furthermore,

$$det(B) \ = \ det(A^{-1})det(C) \ = \ det(A)^{-1}det(C) \ = \ d^{-1}d \ = \ 1 \ .$$

Hence $B \in SL_2(\mathbb{R})$. Thus, $C = AB \in ASL_2(\mathbb{R})$. We have proved that

$$ASL_2(\mathbb{R}) \ = \ \{ \ C \ \Big| \ C \in GL_2(\mathbb{R}) \ \ and \ \ det(C) = d \ \}.$$

Every left coset of $SL_2(\mathbb{R})$ in $GL_2(\mathbb{R})$ has the above description for some $d \in \mathbb{R}, \ d \neq 0$.

There are infinitely many possible choices of $d$ and therefore infinitely many distinct left cosets of $SL_2(\mathbb{R})$ in $GL_2(\mathbb{R})$. Thus, the index of $SL_2(\mathbb{R})$ in $GL_2(\mathbb{R})$ is infinite.

**Section 6.4, Problem 9.**

**Solution.** The group operation in $\mathbb{Q}$ is addition. Of course, $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$. A typical left coset has the form

$$r + \mathbb{Z} \ = \ \{ \ r + n \mid n \in \mathbb{Z} \ \} \ .$$

Notice that if $r$ has denominator $d$ (when expressed in reduced form), then $r + n$ also has denominator $d$ ( when expressed in reduced form). Thus all the elements in the above left coset have exactly the same denominator.

Consider the sequence of rational numbers $r_d = \frac{1}{d}$, where $d$ is any positive integer. The denominator of $r_d$ is $d$. Therefore, if $d, \ d'$ are positive integers and $d \neq d'$, then $r_d$ and $r_{d'}$ are in different left cosets. Thus, the left cosets $r_d + \mathbb{Z}$ and $r_{d'} + \mathbb{Z}$ are different.

It follows that there are infinitely many distinct left cosets of $\mathbb{Z}$ in $\mathbb{Q}$. This means that the index of $\mathbb{Z}$ in $\mathbb{Q}$ is infinite.

**Section 9.3, Problem 8.**

**Solution.** Suppose that $A$ and $B$ are groups and that $\varphi : A \to B$ is an isomorphism. Suppose also that $A$ is a cyclic group. Then $A = \langle a \rangle$ for some $a \in A$. Let $b = \varphi(a)$. Then $b \in B$. Every element in $A$ is of the form $a^k$ for some $k \in \mathbb{Z}$. Since $\varphi$ is surjective, every element in $B$ has the form $\varphi(a^k)$ for some $k \in \mathbb{Z}$. Note that $\varphi(a^k) = \varphi(a)^k = b^k$. Thus, every element in $B$ has the form $b^k$ for some $k \in \mathbb{Z}$. Therefore, $B = \langle b \rangle$. We have proved that if $A$ is a cyclic group, then $B$ is also a cyclic group.

We can apply the above observation to $A = \mathbb{Z}$ and $B = \mathbb{Q}$. We know that $A$ is a cyclic group. In problem set 2, we proved that $B$ is not a cyclic group. Therefore, there cannot be an isomorphism from $A$ to $B$.

**Section 9.3, Problem 9.**

**Solution.** Let $G = \{r \in \mathbb{R} \mid r \neq -1\}$. We define an operation on $G$ by

$$a * b \;=\; a \;+\; b \;+\; ab$$

for all $a, b \in G$. Note that $G$ is a group. This was proved in problem set 1. Define a map $\varphi : G \to U(\mathbb{R})$ by

$$\varphi(a) \;=\; 1 + a$$

Since $a \neq -1$ means that $1 + a \neq 0$, it is clear that $\varphi$ is a bijective map from $G$ to $U(\mathbb{R})$. Furthermore, we have

$$\varphi(a * b) \;=\; \varphi(a + b + ab) \;=\; 1 + a + b + ab \;=\; (1 + a)(1 + b) \;=\; \varphi(a)\varphi(b) \quad .$$

Hence $\varphi$ is a homomorphism. Therefore, $\varphi$ is indeed an isomorphism from $G$ to $U(\mathbb{R})$ and so those two groups are indeed isomorphic.

**Section 9.3, Problem 48.**

**Solution.** By definition,

$$G \times H \;=\; \{\ (g,\ h) \mid g \in G,\ h \in H\ \}, \qquad\qquad H \times G \;=\; \{\ (h,\ g) \mid h \in H,\ g \in G\ \} \quad .$$

The group operations on these sets were defined in class one day. We can define a map $\varphi$ from $G \times H$ to $H \times G$ by $\varphi\big(\ (g,\ h)\ \big) \;=\; (h,\ g)$. Thus, any element $(h,\ g)$ in $H \times G$ is the image under the map $\varphi$ of the element $(g,\ h)$ in $G \times H$, and of no other element in $G \times H$. That is, the map $\varphi$ is a bijective map.

It remains to verify that $\varphi$ is a homomorphism. To see this, suppose that $(g_1,\ h_1)$ and $(g_2,\ h_2)$ are elements of $G \times H$. Then, by definition, $(g_1,\ h_1)(g_2,\ h_2) = (g_1g_2,\ h_1h_2)$. Hence

$$\varphi\big(\ (g_1,\ h_1)(g_2,\ h_2)\ \big) \;=\; \varphi\big(\ (g_1g_2,\ h_1h_2)\ \big) \;=\; (h_1h_2,\ g_1g_2) \;=\; (h_1,\ g_1)(h_2,\ g_2)\ .$$

On the other hand, we have

$$\varphi\big(\ (g_1,\ h_1)\ \big)\varphi\big(\ (g_2,\ h_2)\ \big) \;=\; (h_1,\ g_1)(h_2,\ g_2) \quad .$$

Hence we have $\varphi\big(\ (g_1,\ h_1)(g_2,\ h_2)\ \big)\ =\ \varphi\big(\ (g_1,\ h_1)\ \big)\varphi\big(\ (g_2,\ h_2)\ \big)$. This means that $\varphi$ is indeed a homomorphism from $G\times H$ to $H\times G$. Since $\varphi$ is also bijective, $\varphi$ is an isomorphism. Therefore, $G\times H$ is isomorphic to $H\times G$, as stated.

**Section 9.3, Problem 50.**

**Solution.** Suppose that $A$ and $B$ are abelian groups. This means that if $a_1, a_2 \in A$, then $a_1a_2 = a_2a_1$. Similarly, if $b_1, b_2 \in B$, then $b_1b_2 = b_2b_1$. To show that $A\times B$ is abelian, suppose that $(a_1,\ b_1)$ and $(a_2,\ b_2)$ are arbitrary elements of $A\times B$. Then

$$(a_1,\ b_1)(a_2,\ b_2)\ =\ (a_1a_2,\ b_1b_2)\ =\ (a_2a_1,\ b_2b_1)\ =\ (a_2,\ b_2)(a_1,\ b_1)$$

which proves that $A\times B$ is indeed an abelian group.

Conversely, suppose that $A$ and $B$ are groups and that $A\times B$ is abelian. We will prove that $A$ is abelian. Suppose that $a_1,\ a_2 \in A$. Let $f$ be the identity element in $B$. Consider the elements $(a_1,\ f)$ and $(a_2, f)$ in $A\times B$. Since $A\times B$ is abelian, we have

$$(a_1,\ f)(a_2,\ f)\ =\ (a_2,\ f)(a_1,\ f)\ \ .$$

This means that $(a_1a_2, f) = (a_2a_1, f)$. Equivalently, we have $a_1a_2 = a_2a_1$ and $f = f$. Thus, if $a_1, a_2 \in A$, then $a_1a_2 = a_2a_1$. This proves that $A$ is indeed an abelian group. A similar proof will show that $B$ is also an abelian group.

ADDITIONAL PROBLEMS:

**A:** Let $G = Q_8$. Let $H = \langle -1\rangle$. Let $K = \langle i\rangle$. Both $H$ and $K$ are subgroups of $G$. Find the left cosets of $H$ in $G$. Find the right cosets of $H$ in $G$. Find the left cosets of $K$ in $G$. Find the right cosets of $K$ in $G$.

**Solution.** Since $[G : H] = |G|\big/|H| = 8/2 = 4$, there are four left cosets and four right cosets of $H$ in $G$. However, since $hg = gh$ for all $h \in H$ and $g \in G$, it follows that $H$ is a normal subgroup of $G$. Each left coset will be a right coset. Here they are:

$$H = \{1,\ -1\}, \qquad iH = Hi = \{i,\ -i\}, \qquad jH = Hj = \{j,\ -j\}, \qquad kH = Hk = \{k,\ -k\}\ .$$

Now $K$ is a subgroup of $G$ and $[G:K] = |G|/|K| = 8/4 = 2$. As explained in class, this implies that $K$ is a normal subgroup of $G$, that each left coset is also a right coset, and that there are just two left cosets. Here they are:

$$K = \{1, \ -1, \ i, \ -i\} \qquad \textit{and} \qquad jK = Kj = \{j, \ -j, \ k, \ -k\} \ .$$

**B:** Let $G = S_3$. Let $H = \langle (1\ 2) \rangle$. Find the left cosets of $H$ in $G$. Find the right cosets of $H$ in $G$.

**Solution.** We have $[G:H] = 6/2 = 3$. The three left cosets are

$$H = \{e, \ (1\ 2)\}, \qquad (1\ 3)H = \{(1\ 3), \ (1\ 2\ 3)\}, \qquad (2\ 3)H = \{(2\ 3), \ (1\ 3\ 2)\} \ .$$

There are also three right cosets of $H$ in $G$. They are

$$H = \{e, \ (1\ 2)\}, \qquad H(1\ 3) = \{(1\ 3), \ (1\ 3\ 2)\}, \qquad H(2\ 3) = \{(2\ 3), \ (1\ 2\ 3)\} \ .$$

**C:** Suppose that $G$ is a group and that $c \in G$. Let $H = \{h \in G \mid hc = ch\}$. Thus, $H$ is the set of elements in $G$ which commute with $c$.

(a) Prove that $H$ is a subgroup of $G$.

**Solution.**. Let $e$ be the identity element in $G$. Then $ec = c$ and $ce = c$. Thus, $ec = ce$. It follows that $e \in H$. Now suppose that $h_1, \ h_2 \in H$. This means that $h_1 c = c h_1$ and $h_2 c = c h_2$. It follows that

$$(h_1 h_2)c = h_1(h_2 c) = h_1(c h_2) = (h_1 c)h_2 = (c h_1)h_2 = c(h_1 h_2) \ .$$

Therefore, $h_1 h_2 \in H$. Finally, suppose that $h \in H$. We must verify that $h^{-1} \in H$. We know that $hc = ch$. We also have the following implications:

$$hc = ch \implies h^{-1}hc = h^{-1}ch \implies ec = h^{-1}ch \implies c = h^{-1}ch$$

$$\implies ch^{-1} = h^{-1}chh^{-1} \implies ch^{-1} = h^{-1}ce \implies ch^{-1} = h^{-1}c$$

and this proves that if $h \in H$, then $h^{-1} \in H$. The above observations show that $H$ is a subgroup of $G$.

(b) Suppose that $d \in G$ and that $d$ is conjugate to $c$ in $G$. Prove that the set

$$\{a \in G \mid aca^{-1} = d\}$$

is a left coset of $H$ in $G$.

**Solution.**. We will let $A_d$ denote the set in question. Since we are assuming that $d$ is a conjugate of $c$, the set $A_d$ is nonempty. Pick an element $k \in A_d$. This means that $kck^{-1} = d$. We will prove that

$$A_d = kH$$

Thus, indeed, $A_d$ is a left coset of $H$ in $G$.

We first prove that $kH \subseteq A_d$. To see this, suppose that $a \in kH$. Then $a = kh$, where $h \in H$. Using the definition of $H$, it follows that $hc = ch$. Hence $hch^{-1} = c$. We have

$$aca^{-1} = (kh)c(kh)^{-1} = (kh)c(h^{-1}k^{-1}) = k\big(hch^{-1}\big)k^{-1} = kck^{-1} = d .$$

This shows that $a \in A_d$. Thus, as claimed, we have $kH \subseteq A_d$.

Now we will prove that $A_d \subseteq kH$. To see this, suppose that $a \in A_d$. Then $aca^{-1} = d$. We also have $kck^{-1} = d$. Thus, $aca^{-1} = kck^{-1}$. This equation implies that

$$k^{-1}\big(aca^{-1}\big)k = k^{-1}\big(kck^{-1}\big)k = (k^{-1}k)c(k^{-1}k) = ece = c$$

Therefore, $\big(k^{-1}a\big)c\big(a^{-1}k\big) = c$. Let $h = k^{-1}a$. Note that $h^{-1} = a^{-1}k$. Therefore, combining these remarks, we see that

$$hch^{-1} = c$$

and therefore $hc = ch$. It follows that $h = k^{-1}a \in H$. We have $a = kh$. Therefore, $a = kh \in kH$. We have proved that $A_d \subseteq kH$.

In summary, we have proved that $kH \subseteq A_d$ and that $A_d \subseteq kH$. Therefore, $A_d = kH$, as we stated above.

**D:** Let $G = S_4$. Let $H = \{\ \sigma \in G \mid \sigma(4) = 4\ \}$.

(a) Prove that $H$ is a subgroup of $G$ and that $|H| = 6$.

**Solution.** First we show that $H$ is a subgroup of $G$. Clearly, $e \in H$ because $e(j) = j$ for all $j \in \{1, 2, 3, 4\}$ and hence $e(4) = 4$. Also, if $\sigma_1,\ \sigma_2 \in H$, then $\sigma_1(4) = 4,\ \sigma_2(4) = 4$. Hence

$$\sigma_1 \circ \sigma_2(4) = \sigma_1\big(\sigma_2(4)\big) = \sigma_1(4) = 4$$

This implies that $\sigma_1\sigma_2 = \sigma_1 \circ \sigma_2 \in H$. Thus, $H$ is closed under the group operation for $G$. Finally, suppose that $\sigma \in H$. Then $\sigma(4) = 4$. Thus, the inverse function $\sigma^{-1}$ satisfies $\sigma^{-1}(4) = 4$ and therefore $\sigma^{-1} \in H$. These remarks imply that $H$ is a subgroup of $G$.

The elements of $H$ are of the form $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & 4 \end{pmatrix}$, where $a, b$, and $c$ are distinct elements of the set $\{1, 2, 3\}$. There are six possibilities for $\sigma$. Hence $|H| = 6$.

(b) Suppose that $j \in \{1,\ 2,\ 3,\ 4\}$. Prove that the set $\{\ \sigma \in G \bigm| \sigma(4) = j\ \}$ is a left coset of $H$ in $G$.

**Solution.** There exists an element $g \in G$ such that $g(4) = j$. Pick one such element $g$. We will consider the left coset $gH$. Consider any element $h \in H$. Then $h(4) = 4$ and

$$gh(4) = g \circ h(4) = g\big(h(4)\big) = g(4) = j.$$

Therefore, we have $gH \ = \ \{gh \mid h \in H\} \ \subseteq \ \{\sigma \mid \sigma \in G,\ \sigma(4) = j\}$.

To prove the reverse inclusion, suppose that $\sigma \in G$ satisfies $\sigma(4) = j$. Thus, $\sigma(4) = g(4)$. We can write $\sigma = gu$ by letting $u = g^{-1}\sigma \in G$. We want to prove that $u \in H$. We have

$$u(4) = g^{-1}\sigma(4) = g^{-1} \circ \sigma(4) = g^{-1}\big(\sigma(4)\big) = g^{-1}(j) = 4,$$

the final equality following from the fact that $g(4) = j$ and $g^{-1}$ is the inverse function for $g$. Therefore, $u(4) = 4$, $u \in H$, and $\sigma = gu \in gH$. This proves the inclusion

$$\{\sigma \mid \sigma \in G,\ \sigma(4) = j\} \ \subseteq gH.$$

Combining the two inclusions, we see that $\{\sigma \mid \sigma \in G,\ \sigma(4) = j\} = gH$. Therefore, each left coset of $H$ in $G$ is indeed of the form stated in the problem.

Although it is not part of this question, we will describe the right cosets of $H$ in $G$. The right cosets have the following description. If $g \in G$ and $g^{-1}(4) = j$, where $j \in T$, then

$$Hg \ = \ \{\sigma \mid \sigma \in G,\ \sigma(j) = 4\} \tag{1}$$

The proof is similar to the one for the left cosets. We have $g(j) = 4$. If $h \in H$, then $h(4) = 4$ and so

$$hg(j) = h\big(g(j)\big) = h(4) = 4$$

and so $hg \in \{\sigma \mid \sigma \in G,\ \sigma(j) = 4\}$. Hence, $Hg \subseteq \{\sigma \mid \sigma \in G,\ \sigma(j) = 4\}$. Conversely, suppose $\sigma(j) = 4$, then write $\sigma = ug$ where $u \in G$. We have $4 = \sigma(j) = u\big(g(j)\big) = u(4)$ and hence $u \in H$. Thus, $\sigma \in Hg$. This proves the inclusion $\{\sigma \mid \sigma \in G,\ \sigma(j) = 4\} \subset Hg$. Therefore, the equality (1) is proved.