PROPERTIES OF THE REAL NUMBERS

The set of real numbers is denoted by $\mathbf{R}$. In the properties stated below, it is assumed that $a, b, c, d \in \mathbf{R}$ even if not mentioned.

Algebraic Properties. The set $\mathbf{R}$ is a *"field"*. By definition, this means that there are two binary operations on $\mathbf{R}$ which we call *"addition"* and *"multiplication"*. They are usually denoted by $+$ and $\cdot$, respectively. The following properties are required to hold.

1. Commutative Laws: $a + b = b + a, \qquad a \cdot b = b \cdot a.$

2. Associative Laws: $(a + b) + c = a + (b + c), \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c).$

3. Distributive Law: $a \cdot (b + c) = a \cdot b \ + \ (a \cdot c.$

4. There exists an element 0 in $\mathbf{R}$ such that $a + 0 = a$ for all $a \in \mathbf{R}$.

5. There exists an element 1 in $\mathbf{R}$ such that $a \cdot 1 = a$ for all $a \in \mathbf{R}$.

6. If $a \in \mathbf{R}$, then there exists an element $b \in \mathbf{R}$ such that $a + b = 0$.
(This element b is usually denoted by $-a$.)

7. If $a \in \mathbf{R}$ and $a \neq 0$, then there exists an element $b \in \mathbf{R}$ such that $ab = 1$.
(This element b is usually denoted by b^{-1}.)

8. If $a \in \mathbf{R}$, then $a \cdot 0 = 0$.

Order Properties. There is a relation on the set $\mathbf{R}$ which is denoted by $<$. This relation satisfies the following properties:

1. For $a, b \in \mathbf{R}$, exactly one of the following statements is true:

$$a < b, \qquad a = b, \qquad b < a.$$

2. If $a < b$ and $b < c$, then $a < c$.

3. If $a < b$ and $c < d$, then $a + c < b + d$.

4. If $a < b$ and $0 < c$, then $ac < bc$.

5. If $0 < a$ and $0 < b$ and $a < b$, then $b^{-1} < a^{-1}$.

6. If $a < b$, then $-b < -a$.

7. The Archimedean Law: If $0 < a$ and $0 < b$, then there exists an element $n \in \mathbf{N}$ such that $b < na$.

Completeness of R.

1. Every nonempty subset of $\mathbf{R}$ which is bounded above has a least upper bound.

One very important consequence of the above property is the following theorem.

Theorem 1. If $\{s_n\}$ is a nondecreasing sequence of real numbers which is bounded above, then $\{s_n\}$ is a convergent sequence.

The following properties are also true and can be deduced from the above statements.

2. Every nonempty subset of $\mathbf{R}$ which is bounded below has a greatest lower bound.

Theorem 2. If $\{s_n\}$ is a nonincreasing sequence of real numbers which is bounded below, then $\{s_n\}$ is a convergent sequence.