BASIC THEOREMS ABOUT LIMITS

Theorem 1. Suppose that \( f(x) \) and \( g(x) \) are defined on an interval \((\alpha, \beta)\) and that \( x_0 \in (\alpha, \beta) \). Suppose that \( \lim_{x \to x_0} f(x) = A \) and \( \lim_{x \to x_0} g(x) = B \), where \( A, B \in \mathbb{R} \). Then

\[
\lim_{x \to x_0} (f(x) + g(x)) = A + B \quad \text{and} \quad \lim_{x \to x_0} (f(x)g(x)) = AB .
\]

Furthermore, if \( B \neq 0 \), then \( \lim_{x \to x_0} \left( \frac{f(x)}{g(x)} \right) = \frac{A}{B} \).

Theorem 2. Suppose that \( f(x), g(x), \) and \( h(x) \) are defined on an interval \((\alpha, \beta)\) and that \( x_0 \in (\alpha, \beta) \). Suppose that \( f(x) \leq g(x) \leq h(x) \) for all \( x \in (\alpha, \beta) \). Suppose that \( \lim_{x \to x_0} f(x) = A \) and \( \lim_{x \to x_0} h(x) = A \), where \( A \in \mathbb{R} \). Then \( \lim_{x \to x_0} g(x) = A \).

Theorem 3. Suppose that \( f(x) \) and \( g(x) \) are defined on an interval \((\alpha, \beta)\) and that \( x_0 \in (\alpha, \beta) \). Suppose that \( f(x) \leq g(x) \) for all \( x \in (\alpha, \beta) \). Suppose that \( \lim_{x \to x_0} f(x) = +\infty \) and \( \lim_{x \to x_0} g(x) = B \), where \( A, B \in \mathbb{R} \). Then \( A \leq B \).

Theorem 4. Suppose that \( f(x) \) and \( g(x) \) are defined on an interval \((\alpha, \beta)\) and that \( x_0 \in (\alpha, \beta) \). Suppose that \( f(x) \leq g(x) \) for all \( x \in (\alpha, \beta) \). Suppose that \( \lim_{x \to x_0} f(x) = +\infty \). Then \( \lim_{x \to x_0} g(x) = +\infty \).

A variation on theorems 1, 2, 3, and 4. Suppose that the functions in the above theorems are defined on an interval \((\alpha, +\infty)\) and that \( \lim \) is replaced by \( \lim \). Then the stated conclusions are valid.
Theorem 5. Suppose that $f(x)$ is defined on an interval $(\alpha, \beta)$ and that $x_0 \in (\alpha, \beta)$. Suppose that $\lim_{x \to x_0} f(x) = A$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then

$$\lim_{x \to x_0} F(f(x)) = F(A) .$$

Theorem 6. Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers. Suppose that $\lim_{n \to \infty} s_n = A$ and $\lim_{n \to \infty} t_n = B$, where $A, B \in \mathbb{R}$. Then

$$\lim_{n \to \infty} (s_n + t_n) = A + B \quad \text{and} \quad \lim_{n \to \infty} (s_n t_n) = AB .$$

Furthermore, if $B \neq 0$, then $\lim_{n \to \infty} (s_n / t_n) = A / B$.

Theorem 7. Suppose that $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ are sequences of real numbers. Suppose that

$$r_n \leq s_n \leq t_n$$

for all $n \geq 1$. Suppose that $\lim_{n \to \infty} r_n = A$ and $\lim_{n \to \infty} t_n = A$, where $A \in \mathbb{R}$. Then $\lim_{n \to \infty} s_n = A$.

Theorem 8. Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers. Suppose that $s_n \leq t_n$ for all $n \geq 1$. Suppose that $\lim_{n \to \infty} s_n = A$ and $\lim_{n \to \infty} t_n = B$, where $A, B \in \mathbb{R}$. Then $A \leq B$.

Theorem 9. Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers. Suppose that $s_n \leq t_n$ for all $n \geq 1$. Suppose that $\lim_{n \to \infty} s_n = +\infty$. Then $\lim_{n \to \infty} t_n = +\infty$.

Theorem 10. Suppose that $\{s_n\}$ is a sequence of real numbers with the following property:

$$s_n \leq s_{n+1} \quad \text{for all } n \geq 1 .$$

Suppose also that there exists a real number $M$ such that $s_n \leq M$ for all $n \geq 1$. Then $\{s_n\}$ is a convergent sequence.