BASIC THEOREMS ABOUT LIMITS

Theorem 1. Suppose that $f(x)$ and $g(x)$ are defined on an interval (α, β) and that $x_0 \in (\alpha, \beta)$. Suppose that $\operatorname*{Lim}_{x \to x_0} f(x) = A$ and $\operatorname*{Lim}_{x \to x_0} g(x) = B$, where $A, B \in \mathbb{R}$. Then

$$\operatorname*{Lim}_{x \to x_0} \big(f(x) + g(x)\big) = A + B \qquad \textit{and} \qquad \operatorname*{Lim}_{x \to x_0} \big(f(x)g(x)\big) = AB \quad .$$

Furthermore, if $B \neq 0$, then $\operatorname*{Lim}_{x \to x_0} \big(f(x)/g(x)\big) = A/B$.

Theorem 2. Suppose that $f(x)$, $g(x)$, and $h(x)$ are defined on an interval (α, β) and that $x_0 \in (\alpha, \beta)$. Suppose that

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in (\alpha, \beta)$. Suppose that $\operatorname*{Lim}_{x \to x_0} f(x) = A$ and $\operatorname*{Lim}_{x \to x_0} h(x) = A$, where $A \in \mathbb{R}$. Then $\operatorname*{Lim}_{x \to x_0} g(x) = A$.

Theorem 3. Suppose that $f(x)$ and $g(x)$ are defined on an interval (α, β) and that $x_0 \in (\alpha, \beta)$. Suppose that $f(x) \leq g(x)$ for all $x \in (\alpha, \beta)$. Suppose that $\operatorname*{Lim}_{x \to x_0} f(x) = A$ and $\operatorname*{Lim}_{x \to x_0} g(x) = B$, where $A, B \in \mathbb{R}$. Then $A \leq B$.

Theorem 4. Suppose that $f(x)$ and $g(x)$ are defined on an interval (α, β) and that $x_0 \in (\alpha, \beta)$. Suppose that $f(x) \leq g(x)$ for all $x \in (\alpha, \beta)$. Suppose that $\operatorname*{Lim}_{x \to x_0} f(x) = +\infty$. Then $\operatorname*{Lim}_{x \to x_0} g(x) = +\infty$.

A variation on theorems 1,2, 3, and 4. Suppose that the functions in the above theorems are defined on an interval $(\alpha, +\infty)$ and that $\operatorname*{Lim}_{x \to x_0}$ is replaced by $\operatorname*{Lim}_{x \to +\infty}$. Then the stated conclusions are valid.

Theorem 5. Suppose that $f(x)$ is defined on an interval (α, β) and that $x_0 \in (\alpha, \beta)$. Suppose that $\operatorname*{Lim}_{x \to x_0} f(x) = A$. Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then

$$\operatorname*{Lim}_{x \to x_0} F\big(f(x)\big) = F(A) \quad .$$

Theorem 6. Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers. Suppose that $\operatorname*{Lim}_{n \to \infty} s_n = A$ and $\operatorname*{Lim}_{n \to \infty} t_n = B$, where $A, B \in \mathbb{R}$. Then

$$\operatorname*{Lim}_{n \to \infty} \big(s_n + t_n\big) = A + B \qquad \textit{and} \qquad \operatorname*{Lim}_{n \to \infty} \big(s_n t_n\big) = AB \quad .$$

Furthermore, if $B \neq 0$, then $\operatorname*{Lim}_{n \to \infty} \big(s_n / t_n\big) = A/B$.

Theorem 7. Suppose that $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ are sequences of real numbers. Suppose that

$$r_n \leq s_n \leq t_n$$

for all $n \geq 1$. Suppose that $\operatorname*{Lim}_{n \to \infty} r_n = A$ and $\operatorname*{Lim}_{n \to \infty} t_n = A$, where $A \in \mathbb{R}$. Then $\operatorname*{Lim}_{n \to \infty} s_n = A$.

Theorem 8. Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers. Suppose that $s_n \leq t_n$ for all $n \geq 1$. Suppose that $\operatorname*{Lim}_{n \to \infty} s_n = A$ and $\operatorname*{Lim}_{n \to \infty} t_n = B$, where $A, B \in \mathbb{R}$. Then $A \leq B$.

Theorem 9. Suppose that $\{s_n\}$ and $\{t_n\}$ are sequences of real numbers. Suppose that $s_n \leq t_n$ for all $n \geq 1$. Suppose that $\operatorname*{Lim}_{n \to \infty} s_n = +\infty$. Then $\operatorname*{Lim}_{n \to \infty} t_n = +\infty$.

Theorem 10. Suppose that $\{s_n\}$ is a sequence of real numbers with the following property:

$$s_n \;\leq\; s_{n+1} \qquad \text{for all } n \geq 1 \quad .$$

Suppose also that there exists a real number M such that $s_n \leq M$ for all $n \geq 1$. Then $\{s_n\}$ is a convergent sequence.