Problem 16, page 118: We will just give the answers with partial explanations for the more interesting examples.

(i): The function $f_1$ from $\mathbb{R}$ to $\mathbb{R}$ is both injective and surjective. It is a bijective function from $\mathbb{R}$ to $\mathbb{R}$. Its inverse is given by $g_1(y) = y + 1$ for all $y \in \mathbb{R}$.

(ii): The function $f_2$ from $\mathbb{R}$ to $\mathbb{R}$ is both injective and surjective. It is a bijective function from $\mathbb{R}$ to $\mathbb{R}$. Its inverse is given by $g_2(y) = \sqrt{y}$ for all $y \in \mathbb{R}$.

(iii): The function $f_3$ from $\mathbb{R}$ to $\mathbb{R}$ is surjective, but not injective. The fact that $f_3$ is not injective is seen by noting that the real numbers 0 and 1 are not equal, but the values $f_3(0) = 0$ and $f_3(1) = 0$ are equal. For surjectivity, one must use theorems from calculus. We have

$$\lim_{x \to +\infty} f_3(x) = +\infty, \quad \lim_{x \to -\infty} f_3(x) = -\infty$$

Since $f_3(x)$ is continuous, a theorem in calculus (sometimes called the “intermediate value theorem”) implies that for any $y \in \mathbb{R}$, the equation $f_3(x) = y$ has at least one solution where $x \in \mathbb{R}$. This implies that $f_3$ is surjective.

(iv): One has $f_4(x) = (x - 1)^3$ for all $x \in \mathbb{R}$. The function $f_4$ from $\mathbb{R}$ to $\mathbb{R}$ is both injective and surjective. It is a bijective function from $\mathbb{R}$ to $\mathbb{R}$. Its inverse is given by $g_4(y) = 1 + \sqrt[3]{y}$ for all $y \in \mathbb{R}$.

(v): The function $f_5$ from $\mathbb{R}$ to $\mathbb{R}$ is injective, but not surjective. The fact that $f_5$ is injective is seen by noting that the derivative of $f_5(x)$ is positive for all $x \in \mathbb{R}$ and hence $f_5(x)$ is a strictly increasing function. That fact implies that $f_5(x)$ is injective. Now $f_5(x) > 0$ for all $x \in \mathbb{R}$ and so $f_5(x)$ cannot be a surjective function from $\mathbb{R}$ to $\mathbb{R}$.

(vi): The function $f_6$ from $\mathbb{R}$ to $\mathbb{R}$ is both injective and surjective. It is a bijective function from $\mathbb{R}$ to $\mathbb{R}$. Its inverse is given by $g_6 : \mathbb{R} \to \mathbb{R}$ defined by

$$g_6(y) = \begin{cases} \sqrt{y} & \text{if } y \geq 0 \\ -\sqrt{-y} & \text{if } y \leq 0 \end{cases}$$

Indeed, if $x \geq 0$, then $y = f_6(x) = x^2 \geq 0$ and so $g_6(y) = \sqrt{x^2} = x$ for $x \geq 0$. If $x \leq 0$, then $y = f_6(x) = -x^2 \leq 0$ and so $g_6(y) = -\sqrt{-y} = -\sqrt{x^2} = -|x| = x$ for $x \leq 0$. Thus, $g_6(f_6(x)) = x$ for all $x \in \mathbb{R}$. Therefore, $g_6 \circ f_6 = i_\mathbb{R}$.

Also, for $y \geq 0$, we have $g_6(y) \geq 0$ and so $f_6(g_6(y)) = (\sqrt{y})^2 = y$. For $y \leq 0$, we have $g_6(y) \leq 0$ and so $f_6(g_6(y)) = -(-\sqrt{-y})^2 = -(-y) = y$. Thus, $f_6(g_6(y)) = y$ for all $y \in \mathbb{R}$. Therefore, $f_6 \circ g_6 = i_\mathbb{R}$. 
Problem 17, page 118. We can describe $f \circ g$ as follows:

$$(f \circ g)(x) = \begin{cases} 
  x & \text{if } x < -1 \\
  x & \text{if } -1 \leq x \leq 1 \\
  x & \text{if } x > 1
\end{cases}$$

Thus, $(f \circ g)(x) = x$ for all $x \in \mathbb{R}$. That is, $f \circ g = i_{\mathbb{R}}$.

We can describe $g \circ f$ as follows:

$$(g \circ f)(x) = \begin{cases} 
  x & \text{if } x < -3 \\
  -x - 2 & \text{if } -3 \leq x < -1 \\
  x & \text{if } -1 \leq x \leq 1 \\
  -x + 2 & \text{if } 1 < x \leq 3 \\
  x & \text{if } x > 3
\end{cases}$$

Thus, $g \circ f \neq i_{\mathbb{R}}$. Therefore, $g$ is not an inverse for the function $f$.

The function $f$ is surjective, but not injective. The function $g$ is injective, but not surjective.

A: Suppose that $X, Y,$ and $Z$ are sets. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. Let $h = g \circ f : X \to Z$. Thus, for any $x \in X$, we have $h(x) = g(f(x))$. We use this repeatedly in the following arguments.

We will prove the following propositions.

(i) Proposition. If $f$ and $g$ are injective, then $h$ is injective.

Proof. Assume that $f$ and $g$ are injective. Then, by definition, the following statements are true:

\[
\begin{align*}
  x_1, x_2 \in X & \text{ and } x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \\
  y_1, y_2 \in Y & \text{ and } y_1 \neq y_2 \implies g(y_1) \neq g(y_2)
\end{align*}
\]

Note that if $x_1, x_2 \in X$, then $f(x_1), f(x_2) \in Y$.

Using the above statements, we have

\[
\begin{align*}
  x_1, x_2 \in X & \text{ and } x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \\
  \implies g(f(x_1)) \neq g(f(x_2)) & \implies h(x_1) \neq h(x_2)
\end{align*}
\]

That is, if $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $h(x_1) \neq h(x_2)$. Therefore, $h$ is indeed injective.
(\textit{ii}) \textbf{Proposition.} If $f$ and $g$ are surjective, then $h$ is surjective.

\textbf{Proof.} Assume that $f$ and $g$ are surjective. We want to prove that $h : X \to Z$ is surjective. Let $z \in Z$. Since $g : Y \to Z$ is surjective, it follows that there exists an element $y \in Y$ such that $g(y) = z$. Since $f : X \to Y$ is surjective, there exists an element $x \in X$ such that $f(x) = y$. For such an element $x$, we will have

$$h(x) = g(f(x)) = g(y) = z$$

Hence, we have shown that for any $z \in Z$, there exists an element $x \in X$ such that $h(x) = z$. Therefore, $h : X \to Z$ is indeed surjective.

(\textit{iii}) \textbf{Proposition.} If $f$ and $g$ are bijective, then $h$ is bijective.

\textbf{Proof.} Assume that $f$ and $g$ are bijective. Then $f$ and $g$ are injective and therefore, by part (i), it follows that $h$ is injective. Also, $f$ and $g$ are surjective and therefore, by part (ii), it follows that $h$ is surjective. Thus, $h$ is both injective and surjective. Therefore $h$ is indeed bijective.

(\textit{iv}) \textbf{Proposition.} If $h$ is surjective, then $g$ is surjective.

\textbf{Proof.} Assume that $h : X \to Z$ is surjective. We will show that the function $g : Y \to Z$ is surjective. Let $z$ be any element of $Z$. Since $h$ is assumed to be surjective, there exists an element $x \in X$ such that $h(x) = z$. Let $y = f(x)$. Then $y \in Y$. We have

$$g(y) = g(f(x)) = h(x) = z$$

Hence the element $y \in Y$ has the property that $g(y) = z$. Therefore, we have proved that for any element $z \in Z$, there exists an element $y \in Y$ such that $g(y) = z$. Therefore, $g$ is indeed surjective.

(\textit{v}) \textbf{Proposition.} If $h$ is injective, then $f$ is injective.

\textbf{Proof.} Suppose that $x_1, x_2 \in X$. Assume that $h$ is injective. Therefore, the following statement is true:

$$h(x_1) = h(x_2) \implies x_1 = x_2$$

Note that $f(x_1)$ and $f(x_2)$ are elements of $Y$. We have

$$f(x_1) = f(x_2) \implies g(f(x_1)) = g(f(x_2))$$

$$\implies h(x_1) = h(x_2) \implies x_1 = x_2$$
Thus, we have shown that

\[ f(x_1) = f(x_2) \implies x_1 = x_2 \]

and therefore \( f : X \to Y \) is indeed injective.

We will disprove the following statements by giving a counterexample.

\( (vi) \) **Statement.** If \( h \) is bijective, then \( g \) is injective.

Counterexample: Let \( X = Z = \{1, 2\} \) and let \( Y = \{1, 2, 3\} \).

Define \( f : X \to Y \) by \( f(1) = 1, \ f(2) = 2 \).

Define \( g : Y \to Z \) by \( g(1) = 1, \ g(2) = 2, \ g(3) = 2 \).

Then \( h : X \to Z \) is defined by \( h(1) = g(f(1)) = 1, \ h(2) = g(f(2)) = 2 \).

Clearly, \( h \) is injective. However, \( g \) is not injective because \( g(1) = g(3) \).

\( (vii) \) **Statement.** If \( h \) is bijective, then \( f \) is surjective.

Counterexample: We can use the same counterexample given in part \( (vi) \) of this problem. The function \( h : X \to Z \) is bijective. But the function \( f : X \to Y \) is not surjective because \( 3 \in Y \) and none of the elements \( x \in X \) satisfy \( f(x) = 3 \).

**B:** Define a function \( f : \mathbb{Z}^+ \to \mathbb{Z}^+ \) as follows. For \( k \in \mathbb{Z}^+ \), let

\[ f(k) = \begin{cases} 
3k + 1 & \text{if } k \text{ is odd} \\
\frac{k}{2} & \text{if } k \text{ is even}
\end{cases} \]

Note that the domain and codomain of the function \( f \) is \( \mathbb{Z}^+ \). As explained in class, we define the function \( f^{(n)} \) by iterating this function \( f \) \( n \) times. We will repeatedly use the facts that \( f^{(n+1)} = f \circ f^{(n)} \) and \( f^{(n+1)} = f^{(n)} \circ f \) for all \( n \in \mathbb{Z}^+ \).

(a) For each \( k \in \mathbb{Z}^+ \), consider the following statement:

**P(k):** There exists an \( n \in \mathbb{Z}^+ \) such that \( f^{(n)}(k) = 1 \).

We want to verify \( P(k) \) for \( 1 \leq k \leq 10 \).

We first make a useful observation. Suppose that \( k_1, k_2 \in \mathbb{Z}^+ \) and \( f(k_2) = k_1 \). Then

\[ P(k_1) \implies P(k_2) \]

To justify this, assume that \( P(k_1) \) is true. Thus, there exists an \( n \in \mathbb{Z}^+ \) such that
\[ f^{(n)}(k_1) = 1. \] We therefore have
\[ f^{(n+1)}(k_2) = f^{(n)} \circ f(k_2) = f^{(n)}(f(k_2)) = f^{(n)}(k_1) = 1 \]
Note that \( n + 1 \in \mathbb{Z}^+ \) and, as we just verified, \( f^{(n+1)}(k_2) = 1 \). This shows that \( P(k_2) \) is true.

We will use the above observation repeatedly. First note that \( P(2) \) is true since
\[ f^{(1)}(2) = f(2) = 1. \]
Since \( f(4) = 2 \), the observation implies that \( P(4) \) is true.

Since \( f(1) = 4 \) and \( f(8) = 4 \), \( P(1) \) and \( P(8) \) are both true.

Since \( f(16) = 8 \), \( P(16) \) is true.

Since \( f(5) = 16 \), \( P(5) \) is true.

Since \( f(10) = 5 \), \( P(10) \) is true.

Since \( f(3) = 10 \), \( P(3) \) is true.

Since \( f(6) = 3 \), \( P(6) \) is true.

It remains to verify that \( P(7) \) and \( P(9) \) are true. By iteration, we find
\[ f(7) = 22, \ f(22) = 11, \ f(11) = 34, \ f(34) = 17, \ f(17) = 52, \]
\[ f(52) = 26, \ f(26) = 13, \ f(13) = 40, \ f(40) = 20, \ f(20) = 10 \]
We can now stop because \( P(10) \) has already been verified. Hence \( P(20) \) is true. Hence \( P(40) \) is true. And so on.... This shows that \( P(7) \) is true.

Finally, for \( P(9) \), note that
\[ f(9) = 28, \ f(28) = 14, \ f(14) = 7 \]
Since \( P(7) \) is true, so is \( P(14) \). Hence so is \( P(28) \). Hence \( P(9) \) is true.

(b) Consider the following two statements:
\[ S_1 : \text{For each } n \in \mathbb{Z}^+, \text{ there exists a } k \in \mathbb{Z}^+ \text{ such that } f^{(n)}(k) = 1. \]
\[ S_2 : \text{There exists an } n \in \mathbb{Z}^+ \text{ such that } f^{(n)}(k) = 1 \text{ for all } k \in \mathbb{Z}^+. \]
Statement \( S_1 \) is true. Here is the proof. Suppose that \( n \in \mathbb{Z}^+ \). Let \( k = 2^n \). Obviously, \( k \in \mathbb{Z}^+ \). We will prove that \( f^{(n)}(k) = 1 \). This will prove statement \( S_1 \). We will use mathematical induction. The statement to be proved is
\[ Q(n) : \ f^{(n)}(2^n) = 1 \]
First, consider the base case. Let $n = 1$. Then $f^{(1)}(2^1) = f(2) = 1$, verifying $Q(1)$.

Now, we show that, for any $t \in \mathbb{Z}^+$, $Q(t) \implies Q(t + 1)$. For this purpose, assume that $t \in \mathbb{Z}^+$ and that $Q(t)$ is true. That is, we assume that $f^{(t)}(2^t) = 1$. Note that $2^{t+1}$ is even. Hence, by definition, $f(2^{t+1}) = \frac{1}{2} \cdot 2^{t+1} = 2^t$. We have

$$f^{(t+1)}(2^{t+1}) = f^{(t)} \circ f(2^{t+1}) = f^{(t)}(f(2^{t+1})) = f^{(t)}(2^t) = 1$$

Thus, indeed, we have $f^{(t+1)}(2^{t+1}) = 1$ and so $Q(t + 1)$ is true. We have verified the inductive step.

By the principle of mathematical induction, $Q(n)$ is true for all $n \in \mathbb{Z}^+$. Hence, statement $S_1$ has been proved.

However, statement $S_2$ is false because the negation of that statement is true. We will prove the negation. The negation states that:

**Negation:** For every $n \in \mathbb{Z}^+$, there exists at least one $k \in \mathbb{Z}^+$ such that $f^{(n)}(k) \neq 1$.

**Proof of the negation.** First consider $n = 1$. Take $k = 1$. Then

$$f^{(n)}(k) = f^{(1)}(k) = f(k) = f(1) = 4 \neq 1$$

and hence the negation has been verified when $n = 1$. Now suppose that $n > 1$. Let $m = n - 1 \in \mathbb{Z}^+$. We will use the fact that $f^{(m)}(2^m) = 1$. This is the statement $Q(m)$ which was proved above for all $m \in \mathbb{Z}^+$. We take $k = 2^m$. We then have

$$f^{(n)}(k) = f^{(m+1)}(k) = f \circ f^{(m)}(k) = f(f^{(m)}(k)) = f(f^{(m)}(2^m)) = f(1) = 4 \neq 1$$

and hence the negation has been verified for any $n > 1$.

We have proved the negation of $S_2$. Therefore, $S_2$ is false.

**C:** It is possible to find such functions. Define $f$ and $g$ as follows:

$$f(n) = \begin{cases} n & \text{if } n > 0, \\ 1 & \text{if } n \leq 0 \end{cases}, \quad g(n) = \begin{cases} -1 & \text{if } n > 0, \\ n & \text{if } n \leq 0. \end{cases}$$

Both $f$ and $g$ are nonconstant functions. Notice that $g(n) \leq 0$ for all $n \in \mathbb{Z}$ and hence $(f \circ g)(n) = f(g(n)) = 1$ for all $n \in \mathbb{Z}$. Also, note that $f(n) > 0$ for all $n \in \mathbb{Z}$ and hence $(g \circ f)(n) = g(f(n)) = -1$ for all $n \in \mathbb{Z}$. 