MIDTERM SOLUTIONS – MATH310A – AUTUMN, 2006

QUESTION 1. Here is the truth table for \( P \implies Q \).

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \implies Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>T</td>
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<td>F</td>
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</tbody>
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QUESTION 2. It is assumed in the following proposition that \( a \) and \( b \) are integers.

**Proposition:** If \( ab \) is even, then \( a \) is even or \( b \) is even.

We will prove the contrapositive of the proposition. This will suffice because the contrapositive is logically equivalent to the proposition.

**Contrapositive of the proposition.** If \( a \) and \( b \) are odd, then \( ab \) is odd.

To prove this, assume that \( a \) and \( b \) are odd integers. Then we can write \( a \) and \( b \) in the form \( a = 2p + 1 \), \( b = 2q + 1 \), where \( p, q \in \mathbb{Z} \). It then follows that

\[
(1) \quad ab = (2p + 1)(2q + 1) = 4pq + 2p + 2q + 1 = 2(2pq + p + q) + 1
\]

Since \( p, q \in \mathbb{Z} \), it follows that \( 2pq + p + q \in \mathbb{Z} \). Therefore, by (1), it follows that \( ab \) is indeed odd.

We’ve proved the contrapositive and hence the stated proposition.

QUESTION 3. Here is the statement of Euclid’s Lemma:

**Euclid’s Lemma.** Suppose that \( a, b \in \mathbb{Z} \) and that \( p \) is a prime. If \( p \) divides \( ab \), then \( p \) divides \( a \) or \( p \) divides \( b \).

QUESTION 4. Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a function. We say that \( f \) is “bounded above” if the following statement is true:

**S:** There exists an \( M \in \mathbb{R} \) such that \( f(x) < M \) for all \( x \in \mathbb{R} \).

Here is the negation of \( S \).

For all \( M \in \mathbb{R} \), there exists at least one \( x \in \mathbb{R} \) such that \( f(x) \geq M \).
**QUESTION 5.** The *Strong Mathematical Induction Principle* can be stated as follows:
Suppose that $P(n)$ is a statement involving a general positive integer $n$. Suppose that

(i) $P(1)$ is true

and

(ii) For all positive integers $k$,

\[ P(j) \text{ is true for all positive integers } j \leq k \implies P(k + 1) \text{ is true.} \]

Then $P(n)$ is true for all positive integers $n$.

**QUESTION 6.**

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}^+$ defined by

\[
f(x) = \begin{cases} 
e^x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}
\]

This function $f$ is surjective, but not injective. The fact that $f$ is not injective can be justified by pointing out that $1, -e \in \mathbb{R}$, $1 \neq -e$, but, according to the definition of $f$, we have

\[ f(1) = e^1 = e, \quad f(-e) = -(-e) = e \quad \text{and hence } f(1) = f(-e). \]

The fact that $f$ is surjective is verified by noting that for $y \in \mathbb{R}^+$, $x = -y \in \mathbb{R}$ and satisfies $x < 0$. Thus, for $x = -y$, we have $f(x) = -(-y) = y$. Therefore, $f$ is indeed surjective.

(b) Let $S = \mathbb{R}$. Then $S$ is a subset of $\mathbb{R}$ and, according to Cantor’s theorem stated in class, the set $S$ is not countable.

(c) The two sets are not equal. We have \( \{x \in \mathbb{R} \mid x > 3\} \neq \{x \in \mathbb{R} \mid x^2 > x + 6\} \) because $x = -10$ is a real number and clearly satisfies the inequality $x^2 > x + 6$, but fails to satisfy the inequality $x > 3$. Therefore, -10 is in the second set, but not in the first.