MATH 310 - Homework Assignment 4 (due Friday, February 16th)

From the text: Page 118, problems 16, 17.

A: Suppose that $X,Y$, and $Z$ are sets. Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions. Let $h = g \circ f : X \rightarrow Z$. Prove the following statements.

(i) If $f$ and $g$ are injective, then $h$ is injective.
(ii) If $f$ and $g$ are surjective, then $h$ is surjective.
(iii) If $f$ and $g$ are bijective, then $h$ is bijective.
(iv) If $h$ is surjective, then $g$ is surjective.
(v) If $h$ is injective, then $f$ is injective.

Disprove the following statements by giving a counterexample. For a counterexample, you must specify sets $X,Y$ and $Z$ as well as $f$ and $g$.

(vi) If $h$ is bijective, then $g$ is injective.
(vii) If $h$ is bijective, then $f$ is surjective.

B: Define a function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ as follows. For $k \in \mathbb{Z}^+$, let

$$f(k) = \begin{cases} 
3k + 1 & \text{if } k \text{ is odd} \\
\frac{1}{2}k & \text{if } k \text{ is even}
\end{cases}$$

There is a famous conjecture about this function. We let $f^{(n)} : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ denote the $n$-th iterate of $f$.

**Conjecture:** For all $k \in \mathbb{Z}^+$, there exists an $n \in \mathbb{Z}^+$ such that $f^{(n)}(k) = 1$.

(a) Verify the statement in the conjecture for $1 \leq k \leq 10$. (Try to find an efficient way to do this.)

(b) Consider the following two statements:

$S_1$: For each $n \in \mathbb{Z}^+$, there exists a $k \in \mathbb{Z}^+$ such that $f^{(n)}(k) = 1$.

$S_2$: There exists an $n \in \mathbb{Z}^+$ such that, for all $k \in \mathbb{Z}^+$, $f^{(n)}(k) = 1$.

Prove or disprove each of these statements.

C: Is it possible to find two nonconstant functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $(f \circ g)(n) = 1$ for all $n \in \mathbb{Z}$ and $(g \circ f)(n) = -1$ for all $n \in \mathbb{Z}$.