THE RANGE AND THE NULL SPACE OF A MATRIX

Suppose that $A$ is an $m \times n$ matrix with real entries. There are two important subspaces associated to the matrix $A$. One is a subspace of $\mathbb{R}^m$. The other is a subspace of $\mathbb{R}^n$. We will assume throughout that all vectors have real entries.

THE RANGE OF $A$.

The range of $A$ is a subspace of $\mathbb{R}^m$. We will denote this subspace by $\mathcal{R}(A)$. Here is the definition:

$$\mathcal{R}(A) = \{ Y : \text{there exists at least one } X \text{ in } \mathbb{R}^n \text{ such that } AX = Y \}$$

THEOREM. If $A$ is an $m \times n$ matrix, then $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$.

Proof. First of all, notice that if $Y$ is in $\mathcal{R}(A)$, then $Y = AX$ for some $X$ in $\mathbb{R}^n$. Since $A$ is $m \times n$ and $X$ is $n \times 1$, $Y = AX$ will be $m \times 1$. That is, $Y$ will be in $\mathbb{R}^m$. This shows that the set $\mathcal{R}(A)$ is a subset of $\mathbb{R}^m$.

Now we verify the three subspace requirements. Let $W = \mathcal{R}(A)$.

(a) Let $0_m$ denote the zero vector in $\mathbb{R}^m$ and $0_n$ denote the zero vector in $\mathbb{R}^n$. Notice that $A0_n = 0_m$. Hence $AX = 0_m$ is satisfied by at least one $X$ in $\mathbb{R}^n$, namely $X = 0_n$. Thus, $0_m$ is indeed in $W$ and hence requirement (a) is valid for $W$.

(b) Suppose that $Y_1$ and $Y_2$ are in $W$. This means that each of the matrix equations $AX = Y_1$ and $AX = Y_2$ has at least one solution. Suppose that $X = X_1$ is a vector in $\mathbb{R}^n$ satisfying the first equation. That is, $AX_1 = Y_1$. Suppose that $X = X_2$ is a vector in $\mathbb{R}^n$ satisfying the second equation. That is, $AX_2 = Y_2$. Now consider the matrix equation $AX = Y_1 + Y_2$. Let $X = X_1 + X_2$, a vector in $\mathbb{R}^n$. Then we have

$$AX = A(X_1 + X_2) = AX_1 + AX_2 = Y_1 + Y_2$$

Therefore, $Y_1 + Y_2$ is in $W$. This shows that $W$ is closed under addition and so requirement (b) is valid for $W$.

(c) Suppose that $Y_1$ is in $W$. Let $c$ be any scalar. Since $Y_1$ is in $W$, there exists a vector $X_1$ in $\mathbb{R}^n$ such that $AX_1 = Y_1$. Now consider the matrix equation $AX = cY_1$. Let $X = cX_1$, a vector in $\mathbb{R}^n$. Then we have

$$AX = A(cX_1) = c(AX_1) = cY_1$$
Therefore, \( cY_1 \) is in \( W \). Therefore, \( Y_1 + Y_2 \) is in \( W \). This shows that \( W \) is closed under scalar multiplication and so requirement (c) is valid for \( W \).

We have proved that \( W = \mathcal{R}(A) \) is a subset of \( \mathbb{R}^m \) satisfying the three subspace requirements. Hence \( \mathcal{R}(A) \) is a subspace of \( \mathbb{R}^m \).

**THE NULL SPACE OF \( A \).**

The null space of \( A \) is a subspace of \( \mathbb{R}^n \). We will denote this subspace by \( \mathcal{N}(A) \). Here is the definition:

\[
\mathcal{N}(A) = \{ X : AX = 0_m \}
\]

**THEOREM.** If \( A \) is an \( m \times n \) matrix, then \( \mathcal{N}(A) \) is a subspace of \( \mathbb{R}^n \).

Proof. First of all, notice that if \( X \) is in \( \mathcal{N}(A) \), then \( AX = 0_m \). Since \( A \) is \( m \times n \) and \( AX \) is \( m \times 1 \), it follows that \( X \) must be \( n \times 1 \). That is, \( X \) is in \( \mathbb{R}^n \). Therefore, \( \mathcal{N}(A) \) is a subset of \( \mathbb{R}^n \).

Now we verify the three subspace requirements. Let \( W = \mathcal{N}(A) \).

(a) Notice that \( A0_n = 0_m \). Hence the equation \( AX = 0_m \) is satisfied by \( X = 0_n \). It follows that \( 0_n \) is indeed in \( W \).

(b) Suppose that \( X_1 \) and \( X_2 \) are in \( W \). This means that \( AX_1 = 0_m \) and \( AX_2 = 0_m \). Let \( X = X_1 + X_2 \). Then

\[
AX = A(X_1 + X_2) = AX_1 + AX_2 = 0_m + 0_m = 0_m
\]

Therefore, \( X = X_1 + X_2 \) is in \( W \). This shows that \( W \) is closed under addition and so requirement (b) is valid for \( W \).

(c) Suppose that \( X_1 \) is in \( W \). Let \( c \) be any scalar. Since \( X_1 \) is in \( W \), we have \( AX_1 = 0_m \). Let \( X = cX_1 \). Then

\[
AX = A(cX_1) = c(AX_1) = c0_m = 0_m
\]

Therefore, \( X = cX_1 \) is in \( W \). This shows that \( W \) is closed under scalar multiplication and so requirement (c) is valid for \( W \).

We have proved that \( W = \mathcal{N}(A) \) is a subset of \( \mathbb{R}^m \) satisfying the three subspace requirements. Hence \( \mathcal{N}(A) \) is a subspace of \( \mathbb{R}^n \).

**THE DIMENSION THEOREM:** If \( A \) is an \( m \times n \) matrix, then \( \dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A)) = n \).