A. Do parts (b), (c), and (d) of problem 1 on pages 13-14.

(b) **SOLUTION:** The identity in question is

\[ P(n) : \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \]

To verify P(1), note that the left side just has one term, namely \(1^3 = 1\) and so the left side is equal to 1. The right side is equal to \(1^22^2/4 = 1\). The equality does hold.

Now assume that \(P(n)\) is true. We then have

\[ \sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 \]

\[ = \frac{1}{4}(n^2(n+1)^2 + 4(n+1)^3) = \frac{1}{4}((n+1)^2)(n^2 + 4(n+1)) = \frac{1}{4}((n+1)^2)(n^2 + 4n + 4) \]

\[ = \frac{1}{4}((n+1)^2)(n+2)^2 = \frac{(n+1)^2(n+1+1)^2}{4} \]

which is \(P(n+1)\). We have shown that \(P(n)\) implies \(P(n+1)\).

Therefore, by Mathematical Induction, we have proved that \(P(n)\) is true for all \(n \geq 1\).

(c) **SOLUTION:** We want to prove

\[ P(n) : \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = n + 1 \]

for all \(n \geq 1\). To verify P(1), note that for \(n = 1\), the product has exactly one factor, namely the factor

\[ 1 + \frac{1}{1} = 2 \]

and so the left side of \(P(1)\) is equal to 2. The right side is also equal to 2 when \(n = 1\).
Now assume that $P(n)$ is true for some $n \geq 1$. We then have
\[
\prod_{k=1}^{n+1} \left(1 + \frac{1}{k}\right) = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) \cdot \left(1 + \frac{1}{n+1}\right)
\]
\[
= (n + 1) \cdot \left(1 + \frac{1}{n+1}\right) = \frac{n+2}{n+1} = n + 2 = n + 1 + 1
\]
which is the statement $P(n+1)$. We have proved that $P(n)$ implies $P(n+1)$.

By the Principle of Mathematical Induction, we have proved that $P(n)$ is true for all $n \geq 1$.

(d) **SOLUTION:** We want to prove
\[
P(n) \quad \sum_{k=1}^{n} (2k - 1) = n^2
\]
for all $n \geq 1$. To verify $P(1)$, note that for $n = 1$, the left side has just one term, namely $2 \cdot 1 - 1 = 1$. The right side is $1^2 = 1$. The equality holds. Hence $P(1)$ is true.

Now assume that $P(n)$ is true for some value of $n \geq 1$. We must verify that $P(n+1)$ is also true. To do this, note that
\[
\sum_{k=1}^{n+1} (2k - 1) = \sum_{k=1}^{n} (2k - 1) + (2 \cdot (n + 1) - 1) = n^2 + (2 \cdot (n + 1) - 1)
\]
\[
= n^2 + 2n + 2 - 1 = n^2 + 2n + 1 = (n + 1)^2
\]
which is the statement $P(n+1)$. We have proved that $P(n)$ implies $P(n+1)$.

By the Principle of Mathematical Induction, we have proved that $P(n)$ is true for all $n \geq 1$.

B. This problem concerns the Fibonacci sequence $\{F_n\}$ defined on page 11. By definition, $F_1 = 1, F_2 = 1,$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Prove the following statement:
\[
F_{n+1}^2 - F_{n+2}F_n = (-1)^n
\]
for all $n \geq 1$. 

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SOLUTION: First of all, if $n = 1$, we can verify the stated equality. Indeed, we have $F_1 = F_2 = 1$ and $F_3 = 2$ and hence

$$F_2^2 - F_3F_1 = 1^2 - 2 \cdot 1 = -1 = (-1)^1.$$ 

Suppose that $m \geq 1$. Let us make the assumption that the statement in question is true for $n = m$. That is, we assume that

$$F_{m+1}^2 - F_{m+2}F_m = (-1)^m.$$

We prove the corresponding identity when $n = m + 1$ as follows. We have

$$F_{m+2}^2 - F_{m+3}F_{m+1} = F_{m+2}^2 - (F_{m+2} + F_{m+1})F_{m+1} = F_{m+2}^2 - F_{m+2}F_{m+1} - F_{m+1}F_{m+1}$$

$$= F_{m+2}(F_{m+2} - F_{m+1}) - F_{m+1}^2 = F_{m+2}F_m - F_{m+1}^2$$

$$= - (F_{m+1}^2 - F_{m+2}F_m) = -(-1)^m = (-1)^{m+1}$$

and so we have

$$F_{m+2}^2 - F_{m+3}F_{m+1} = (-1)^{m+1}$$

which is the desired formula when $n = m + 1$. In the above calculation, we have used the two facts

$$F_{m+3} = F_{m+2} + F_{m+1} \quad \text{and} \quad F_{m+2} - F_{m+1} = F_m$$

which follow from the definition of the Fibonacci sequence.

The Principle of Mathematical Induction allows us to now conclude that the identity in question is true for all $n \geq 1$.

C. Prove statements (1), (2), (3), and (4) in Section 2.1 on page 22.

SOLUTION: Here are the proofs. It is assumed that $a, b, c, x,$ and $y$ are integers.

(1) We have $0 = a \cdot 0$ and hence $a|0$. We have $a = a \cdot 1$ and hence $a|a$. We have $b = 1 \cdot b$ and hence $1|b$. Finally, we have $b = (-1) \cdot (-b)$ and hence $-1|b$.

We have used the facts that $0, 1, -1 \in \mathbb{Z}$ and if $b \in \mathbb{Z}$, then $-b \in \mathbb{Z}$ too.

(2) Assume that $a|b$ and $b|c$. Then $b = aq$ where $q \in \mathbb{Z}$ and $c = br$ where $r \in \mathbb{Z}$. Therefore,

$$c = br = (aq)r = a(gr).$$
Since $qr \in \mathbb{Z}$, it follows that $a\mid c$, as we wanted to prove.

(3) Assume that $a\mid b$ and $a\mid c$. Then $b = aq$ and $c = ar$, where $q, r \in \mathbb{Z}$. Therefore,
\[
 bx + cy = (aq)x + (ar)y = a(qx) + a(ry) = a(qx + ry) .
\]
Since $q, x, r, y$ are integers, $qx + ry$ is also an integer. It follows that $bx + cy$ is indeed divisible by $a$. We have proved that $a\mid (bx + cy)$ for all $x, y \in \mathbb{Z}$.

(4) Assume that $a\mid b$ and $b \neq 0$. We have $b = aq$ where $q \in \mathbb{Z}$. Since $b \neq 0$, it follows that $q \neq 0$. Thus, $q$ is a nonzero integer. It follows that $q \geq 1$ or $q \leq -1$. Thus, $|q| \geq 1$. Therefore,
\[
 |b| = |aq| = |a| \cdot |q| \geq |a| \cdot 1 = |a|
\]
and so we indeed have $|b| \geq |a|$.

D. Suppose that $n$ is composite. Prove that there exists at least one prime $p$ such that $p\mid n$ and $p \leq \sqrt{n}$.

**SOLUTION:** Since $n$ is assumed to be composite, we know that $n = ab$, where $a, b \in \mathbb{Z}$ and $a > 1$ and $b > 1$. We either have $a \geq b$ or $b \geq a$. Since $ab = ba$, we can assume that $a \geq b$ without any loss of generality. We then have
\[
 n = ab \geq bb = b^2
\]
and so we have $b \leq \sqrt{n}$.

Since $b$ is an integer and $b > 1$, we can apply a proposition proved in class to conclude that there exists at least one prime $p$ such that $p\mid b$. Note that we have
\[
 p \leq b \leq \sqrt{n} .
\]
Furthermore, $b = pq$, where $q \in \mathbb{Z}$, and hence $n = ab = aq = pc$, where $c = aq \in \mathbb{Z}$. Therefore, $p\mid n$.

We have proved the existence of a prime $p$ such that $p \leq \sqrt{n}$ and $p\mid n$, as we wanted to prove.

E. Find the closest integer to 301 which is a prime.
SOLUTION: First, as mentioned in class, 301 is not itself prime. We will test odd integers which differ by 2,4,6,... from 301 until we find a prime. Apart from the prime 2, any prime must be an odd integer. We have

\[303 = 3 \cdot 101, \quad 299 = 13 \cdot 23,\]
\[305 = 5 \cdot 61, \quad 297 = 3 \cdot 99,\]

and so 297, 299, 303, and 305 are composite. However, 307 is prime. To verify this, note that \(18^2 = 324 > 307\) and so \(\sqrt{307} < 18\). According to problem D, if 307 is composite, it must have a prime divisor which is \(< 18\). Those primes are 2, 3, 5, 7, 11, 13, and 17. One checks quickly that

\[2 \nmid 307, \quad 3 \nmid 307, \quad 5 \nmid 307, \quad 7 \nmid 307, \quad 11 \nmid 307, \quad 13 \nmid 307, \quad 17 \nmid 307,\]

and therefore, we can conclude that 307 is a prime. It follows that the closest prime to 301 is the prime 307. Note that 295 is equally close, but is clearly not a prime.

F. Suppose that \(n, a,\) and \(b\) are integers. Prove or disprove the following statement: \(\text{If } n \mid ab, \text{ then } n \mid a \text{ or } n \mid b.\)

SOLUTION: As mentioned in class, we consider the reworded statement:

\[\text{If } n, a, b \in \mathbb{Z} \text{ and } n \mid ab, \text{ then } n \mid a \text{ or } n \mid b.\]

The statement is false. Consider \(n = 4, a = 2, b = 2.\) Then \(ab = 4\) and it is true that \(n \mid ab.\) However, it is not true that \(n \mid a\) or \(n \mid b.\)

G. Prove or disprove the following statement: \(\text{If } n \text{ is prime, then } 2^n - 1 \text{ is prime.}\)

SOLUTION: The statement is false. Let \(n = 11.\) Then \(n\) is a prime. However,

\[2^{11} - 1 = 2048 - 1 = 2047 = 23 \cdot 89\]

and hence \(2^n - 1\) is not prime when \(n = 11.\)