Galois Theory Problems

1. Suppose that $F$ and $K$ are subfields of $\mathbb{C}$, that $F \subset K$, and that $[K : F] = 2$. Prove that $K$ is a Galois extension of $F$.

2. Determine the Galois groups for the splitting fields of the following polynomials over the specified fields.
   
   (a) $x^3 - x - 1$ over $\mathbb{Q}$, over $\mathbb{Q}(\sqrt{-23})$, over $\mathbb{Q}(\sqrt{23})$.
   
   (b) $x^8 - 1$ over $\mathbb{Q}$, over $\mathbb{Q}(i)$.
   
   (c) $x^3 - x^2 - 4$ over $\mathbb{Q}$.
   
   (d) $x^3 - x^2 - 2x + 1$ over $\mathbb{Q}$.

3. Let $K = \mathbb{Q}(\omega)$, where $\omega = \cos(\frac{2\pi}{17}) + \sin(\frac{2\pi}{17})i$. Prove that $K$ contains a unique subfield $L$ such that $[L : \mathbb{Q}] = 8$. Prove that $L$ is a Galois extension of $\mathbb{Q}$. Find an element $\beta \in L$ such that $L = \mathbb{Q}(\beta)$.

4. Suppose that $\kappa, \lambda \in \mathbb{Q}$. Assume that $K = \mathbb{Q}(\kappa)$ and $L = \mathbb{Q}(\lambda)$ are Galois extensions of $\mathbb{Q}$ and that $[K : \mathbb{Q}] = [L : \mathbb{Q}] = 3$. Furthermore, assume that $K \neq L$. Let $M = \mathbb{Q}(\kappa, \lambda)$. Prove that $M$ is a Galois extension of $\mathbb{Q}$ and that $\text{Gal}(M/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

5. Prove the existence of a Galois extension $K$ of $\mathbb{Q}$ such that $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/14\mathbb{Z}$.

6. This question concerns three subfields $F, K,$ and $L$ of $\mathbb{C}$. Let $F$ be the splitting field over $\mathbb{Q}$ for the polynomial $f(x) = (x^2 + x + 1)(x^2 + x - 1)$. Let $K = \mathbb{Q}(\beta)$, where $\beta$ is the real root of the polynomial $g(x) = x^3 + 3x + 1$. Let $L$ denote the splitting field over $\mathbb{Q}$ for the polynomial $g(x)$. Determine the field $F \cap K$ as precisely as you can. Determine the field $F \cap L$ as precisely as you can.

7. Suppose that $K$ is a finite, Galois extension of $\mathbb{Q}$ and that $\text{Gal}(K/\mathbb{Q}) \cong S_4$. Prove that there exists a polynomial $g(x) \in \mathbb{Q}[x]$ such that $g(x)$ has degree 4 and $K$ is the splitting field for $g(x)$ over $\mathbb{Q}$.

8. Suppose that $K$ is a finite, Galois extension of $\mathbb{Q}$ and that $\text{Gal}(K/\mathbb{Q}) \cong S_4$. Prove that there exists an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ such that all three roots of $f(x)$ are in the field $K$. 