Some Ring Theory Problems

1. Suppose that $I$ and $J$ are ideals in a ring $R$. Assume that $I \cup J$ is an ideal of $R$. Prove that $I \subseteq J$ or $J \subseteq I$.

2. Find an example of an integral domain $R$ and two ideals $I$ and $J$ of $R$ with the following properties: Both $I$ and $J$ are principal ideals of $R$, but $I + J$ is not a principal ideal of $R$.

3. Suppose that $R$ is a commutative ring with identity and that $K$ is an ideal of $R$. Let $R' = R/K$. The correspondence theorem gives a certain one-to-one correspondence between the set of ideals of $R$ containing $K$ and the set of ideals of $R'$. If $I$ is an ideal of $R$ containing $K$, we let $I'$ denote the corresponding ideal of $R'$. Show that if $I$ is principal, then so is $I'$.

4. Suppose that $R$ is an integral domain. Suppose that $I$ and $J$ are ideals in $R$ and that $I = (b)$ where $b \in R$. Prove that $I + J = R$ is and only if $b + I$ is a unit in the ring $R/J$.

5. Suppose that $R$ is an integral domain and that $a, b \in R$. We say that $a$ and $b$ are “relatively prime” if $(a) + (b) = R$. Suppose that $c \in R$. Assume that $a$ and $b$ are relatively prime and that $a|bc$ in $R$. Prove that $a|c$ in $R$.

6. Suppose that $R$ is a PID. Suppose that $a, b$ are nonzero elements of $R$ and that they are relatively prime. Prove that $(a) \cap (b) = (ab)$. Furthermore, consider the map

\[ \varphi : R/(ab) \longrightarrow R/(a) \times R/(b) \]

defined by $\varphi(r + (ab)) = (r + (a), r + (b))$ for all $r \in R$. Prove that $\varphi$ is a well-defined map and that it is a ring isomorphism. (This result is often referred to as the Chinese Remainder Theorem.)

7. Suppose that $R = \mathbb{Z}[\sqrt{2}]$. Suppose that $M_1$ and $M_2$ are maximal ideals of $R$. True or False: If the rings $R/M_1$ and $R/M_2$ are isomorphic, then $M_1 = M_2$. If true, give a proof. If false, give a counterexample.

8. Give an explicit example of an injective ring homomorphism from $\mathbb{Z}/5\mathbb{Z}$ to $\mathbb{Z}/20\mathbb{Z}$. No justification of your answer is needed.

9. Consider the ring $R = \mathbb{Q}[x]/I$, where $I = (x^2 - x)$. Show that $\beta = x + I$ is an idempotent element in $R$, but that $\beta \neq 0_R$ and $\beta \neq 1_R$. Find an idempotent element in $R$ which is not
equal to $0_R$, $1_R$ or $\beta$. Prove that $R \cong \mathbb{Q} \times \mathbb{Q}$. (It may be helpful to review the exercises about idempotents.)

10. This question concerns ring homomorphisms $\varphi$ from a ring $R$ to a ring $S$. In each part of this question, give an example of $R$, $S$, and $\varphi$ satisfying the stated requirements. No explanations are needed. You must specify $R$, $S$, and $\varphi$ precisely.

(a) $R$ is a field, $S$ is not a field, and $\varphi$ is injective.
(b) $R$ and $S$ are integral domains, $\varphi$ is surjective, but not injective.
(c) $R$ is a noncommutative ring, $S$ is an integral domain, and $\varphi$ is surjective.

11. Give a specific example of a prime ideal in the ring $\mathbb{Q}[x]$ which is not a maximal ideal.

12. This question concerns the ring $\mathbb{Z}[i]$. The integer 11213 is a prime number. Furthermore, it turns out that $11213 = 82^2 + 67^2$. You may use these facts in this question without verifying them.

(a) Find a maximal ideal $I$ in the ring $\mathbb{Z}[i]$ which contains 11213. Explain why your ideal $I$ is actually a maximal ideal in $\mathbb{Z}[i]$.
(b) Find all of the irreducible elements $\alpha$ in $\mathbb{Z}[i]$ which divide 11213 in that ring.
(c) Prove that $\mathbb{Z}[i]/I$ is isomorphic to $\mathbb{Z}/11213\mathbb{Z}$. 