FINAL EXAM FOR MATH 308 - AUTUMN, 2004

INSTRUCTIONS: Please read the questions carefully. Show your work clearly and completely. Your explanations should be understandable and convincing. If $A$ is any matrix, then $\mathcal{N}(A)$ denotes the null space of $A$ and $\mathcal{R}(A)$ denotes the range of $A$.

QUESTION 1. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & 3 \\ 1 & 1 & 3 \end{bmatrix}$. Let $V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and $V_3 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

(a) Compute $\text{Det}(A)$.
(b) Find $A^{-1}$.
(c) There is a unique matrix $B$ such that $BA = \begin{bmatrix} 4 & 9 & -1 \end{bmatrix}$. Find this matrix $B$.
(d) Suppose that $C$ is a $3 \times 3$ matrix. Suppose that you are told that $C$ is row-equivalent to $A$. Without knowing anything more about the matrix $C$, can you determine the number of solutions $X$ to the matrix equation $CX = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$? Explain your answer briefly.
(e) Suppose that $F$ is a $3 \times 3$ matrix. Suppose that you are told that $V_1, V_2,$ and $V_3$ are eigenvectors for the matrix $F$, and that the corresponding eigenvalues are $2, 1, \text{and } 5$, respectively. Without knowing anything more about the matrix $F$, can you determine $F$? If so, what is $F$?

QUESTION 2. For each of the following matrices $A$, find the eigenvalues. For each eigenvalue, find a basis for the corresponding eigenspace. Determine if the matrix $A$ is diagonalizable. If $A$ is diagonalizable, then find an invertible matrix $T$ and a diagonal matrix $D$ such that $A = TDT^{-1}$.

(a) $A = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, (b) $A = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 4 & -5 \\ 0 & 0 & 3 \end{bmatrix}$. 
QUESTION 3. In each part of this question, find a specific example of a matrix satisfying all of the stated requirements if it is possible. If no such example exists, explain why.

(a) Give an example (if possible) of a $3 \times 3$ matrix $A$ such that each of the matrix equations

$$AX = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad AX = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has at least one solution, but the matrix equation $AX = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ has no solutions.

(b) Give an example (if possible) of a $3 \times 4$ matrix $B$ with the following two properties:

(i) $\text{rank}(B) = 2$ and (ii) the matrix equation $BX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solutions.

(c) Give an example (if possible) of a $2 \times 2$ matrix $C$ which is singular, but not diagonalizable.

QUESTION 4.

(a) Let $W = \text{Sp}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$. Find a $3 \times 3$ matrix $P$ such that $\mathcal{N}(P) = W$.

(Note: In parts (b), (c), and (d) of this question, use the matrix $P$ that you found in part (a). If you were unable to do part (a), then just choose any matrix $P$ such that $\text{rank}(P) = 1$.)

(b) Find a basis for $\mathcal{R}(P)$.

(c) Let $U$ be any nonzero column in the matrix $P$ that you found in part (a). Verify that $U$ is an eigenvector for $P$.

(d) Determine the eigenvalues of the matrix $P$ and their algebraic multiplicities.

QUESTION 5. Suppose that $A$ is a $2 \times 2$ matrix and that the characteristic polynomial of $A$ is $p(t) = t^2 - 4$. This information does not determine the matrix $A$. There are many such matrices. However, this information is sufficient to determine $A^6$. What is the matrix $A^6$? Explain your answer carefully.