THE LEFT AND RIGHT COSET DECOMPOSITIONS

We assume that $G$ is a group and $H$ is a subgroup of $G$.

**Definition:** Suppose that $a \in G$. The set $aH = \{ah \mid h \in H\}$ is called the *left coset* of $H$ for $a$. The set $Ha = \{ha \mid h \in H\}$ is called the *right coset* of $H$ for $a$.

**Basic Properties:**

1. If $h \in H$, then $hH = Hh = H$. Thus, $H$ is both a left coset and a right coset for $H$.

2. If $a \in G$, then there is a bijection between $H$ and $aH$. Thus, every left coset of $H$ in $G$ has the same cardinality as $H$. The same statements are true for the right cosets of $H$ in $G$.

3. Suppose that $a \in G$. If $b \in aH$, then $bH = aH$. Similarly, if $b \in Ha$, then $Hb = Ha$.

4. If two left cosets of $H$ in $G$ intersect, then they coincide. If two right cosets of $H$ in $G$ intersect, then they coincide.

5. Every element of $G$ belongs to exactly one left coset of $H$ in $G$. Every element of $G$ belongs to exactly one right coset of $H$ in $G$. Thus, $G$ is the disjoint union of the distinct left cosets of $H$ in $G$. Also, $G$ is the disjoint union of the distinct right cosets of $H$ in $G$.

6. The set of left cosets of $H$ in $G$ (denoted by $G/H$) has the same cardinality as the set of right cosets of $H$ (denoted by $H\backslash G$). If these sets are finite, their cardinality is denoted by $[G : H]$.

7. (The order-index equation) If $G$ is finite, then $|G| = [G : H] |H|$.

8. If $a, b \in G$, we will write $a \equiv_L b \pmod{H}$ if $a^{-1}b \in H$. We refer to this relation on $G$ as “left congruence modulo $H$”. It is an equivalence relation on $G$. The equivalence classes are precisely the left cosets of $H$ in $G$. If $a \in G$, then the equivalence class for $a$ under left congruence modulo $H$ is the left coset $aH$.

9. If $a, b \in G$, we write $a \equiv_R b \pmod{H}$ if $ab^{-1} \in H$. We refer to this relation on $G$ as “right congruence modulo $H$”. Similar statements to those in (8) are valid.

10. Suppose that $a \in G$. Then $aH = Ha$ if and only if $aHa^{-1} = H$. 

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