Upper and Lower Bounds on Norms of Functions of Matrices

Given an n by n matrix A, find a set $S \subset \mathbf{C}$ that can be associated with A to give more information than the spectrum alone can provide about the 2-norm of functions of A.

- Field of values:

$$W(A) = \{\langle Aq, q\rangle : \ \langle q, q\rangle = 1\}.$$

- ϵ-pseudospectrum:

$$\sigma_\epsilon(A) = \{z \in \mathbf{C} : \ z \text{ is an eigenvalue of } A + E \text{ for some } E \text{ with } \|E\| < \epsilon\}.$$

- Polynomial numerical hull of degree k:

$$\mathcal{H}_k(A) = \{z \in \mathbf{C} : \ \|p(A)\| \geq |p(z)| \ \forall p \in \mathcal{P}_k\}.$$

Find a set S and scalars m and M with M/m of moderate size such that for all polynomials (or analytic functions) p:

$$m \cdot \sup_{z \in S} |p(z)| \leq \|p(A)\| \leq M \cdot \sup_{z \in S} |p(z)|.$$

- $S = \sigma(A)$, $m = 1$, $M = \kappa(V)$.

  If A is normal then $m = M = 1$, but if A is nonnormal then $\kappa(V)$ may be huge. Moreover, if columns of V have norm 1, then $\kappa(V)$ is close to smallest value that can be used for M.

- If A is nonnormal, might want S to contain more than the spectrum. BUT...

  If S contains more than $\sigma(A)$, must take $m = 0$ since if p is minimal polynomial of A then $p(A) = 0$ but $p(z) = 0$ only if $z \in \sigma(A)$.

- How to modify the problem?

$$m \cdot \sup_{z \in S} |p_{r-1}(z)| \le \|p(A)\| \le M \cdot \sup_{z \in S} |p(z)|$$

- If degree of minimal polynomial is r, then any $p(A) = p_{r-1}(A)$ for a certain $(r-1)$st degree polynomial – the one that matches p at the eigenvalues, and whose derivatives of order up through $t-1$ match those of p at an eigenvalue corresponding to a t by t Jordan block.

- The largest set S where above holds with $m = 1$ is called the **polynomial numerical hull of degree $\mathbf{r-1}$**. In general, however, we do not know good values for M ($<< \kappa(V)$).

Guess a set S. For each p consider

$$\text{inf}\{\|f\|_{\mathcal{L}^\infty(S)} : \ f(A) = p(A)\}. \qquad (*)$$

Find scalars m and M such that for all p:

$$m \cdot (*) \leq \|p(A)\| \leq M \cdot (*).$$

- $f(A) = p(A)$ if $f(z) = p_{r-1}(z) + \chi(z)h(z)$ for some $h \in H^\infty(S)$. Here χ is the minimal polynomial (of degree r) and p_{r-1} is the polynomial of degree $r-1$ satisfying $p_{r-1}(A) = p(A)$.

- (*) is a **Pick-Nevanlinna interpolation problem**.

Given $S \subset \mathbf{C}$, $\lambda_1, \ldots, \lambda_n \in S$, and $w_1, \ldots, w_n$, find

$$\inf\{\|f\|_{\mathcal{L}^\infty(S)} : \ f(\lambda_j) = w_j, \ j = 1, \ldots, n\}.$$

- If S is the open unit disk, then infimum is achieved by a function $\tilde{f}$ that is a scalar multiple of a finite **Blaschke product**:

$$\tilde{f}(z) = \mu \prod_{k=0}^{n-1} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \quad |\alpha_k| < 1$$
$$= \mu \frac{\gamma_0 + \gamma_1 z + \ldots + \gamma_{n-1} z^{n-1}}{\bar{\gamma}_{n-1} + \bar{\gamma}_{n-2} z + \ldots + \bar{\gamma}_0 z^{n-1}}.$$

- Using second representation, Glader and Lindström showed how to compute $\tilde{f}$ and $\|\tilde{f}\|_{\mathcal{L}^\infty(\mathcal{D})}$ by solving a simple eigenvalue problem.

- If S is a simply connected set (such as the field of values) map it conformally to $\overline{\mathcal{D}}$ and solve the problem there.

How do we compute the minimal-norm interpolating function $\tilde{f}$?

As noted earlier, it has the form

$$\tilde{f}(z) = \mu \frac{\gamma_0 + \gamma_1 z + \ldots + \gamma_{n-1} z^{n-1}}{\bar{\gamma}_{n-1} + \bar{\gamma}_{n-2} z + \ldots + \bar{\gamma}_0 z^{n-1}},$$

and satisfies $\tilde{f}(\lambda_j) = p(\lambda_j)$, $j = 1, \ldots, n$.

Let V be the Vandermonde matrix for $\lambda_1, \ldots, \lambda_n$:

$$V^T = \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{pmatrix}.$$

If $\gamma = (\gamma_0, \ldots, \gamma_{n-1})^T$, and Π is the permutation matrix with 1's on its skew diagonal, then these conditions are:

$$V^{-T} p(\Lambda) V^T \Pi \bar{\gamma} = \mu \gamma.$$

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Glader and Lindström showed that there is a real scalar μ for which this equation has a nonzero solution vector γ and that the largest such μ is $\|\tilde{f}\|_{\mathcal{L}^\infty(\mathcal{D})}$. Equate real and imaginary parts to get a $2n$ by $2n$ eigenvalue problem. $\square$

$$\boxed{V^{-T} p(\Lambda) V^T \Pi \bar{\gamma} = \mu \gamma}$$

- **Companion matrices** (with eigenvalues in $\mathcal{D}$) have the form $V \Lambda V^{-1}$ where V is same Vandermonde matrix as above. Therefore above equation becomes

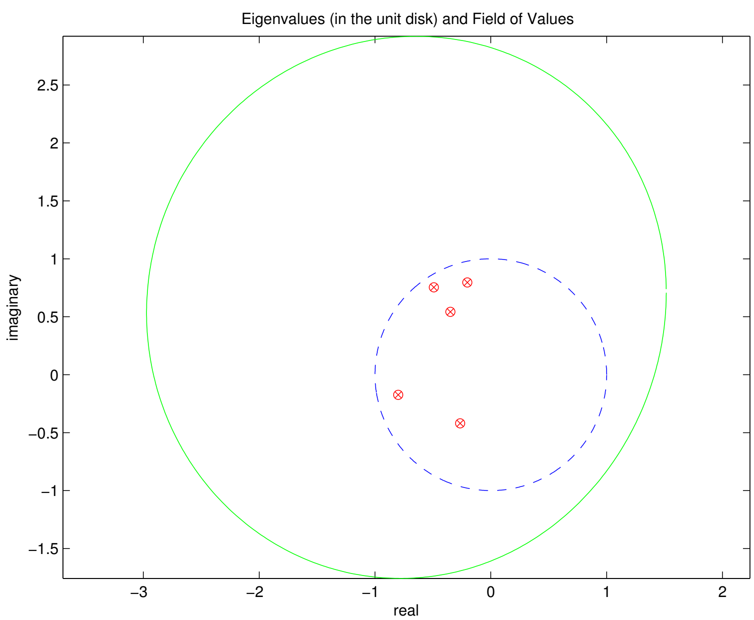
$$(p(A))^T \Pi \bar{\gamma} = \mu \gamma \Rightarrow \|p(A)\| \geq \mu;$$

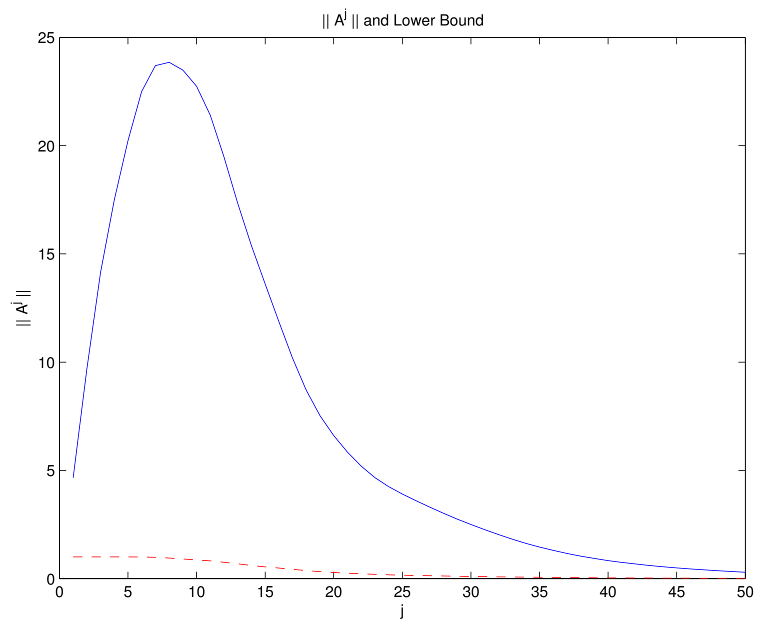
i.e., $\forall p$

$$\|p(A)\| \geq \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(A) = p(A)\},$$

so $m = 1$. In general, do not have good values for M.

Example: Companion matrix with 5 random eigenvalues in the unit disk. $\|A^j\|$ and lower bound.





$$\boxed{V^{-T} p(\Lambda) V^T \Pi \bar{\gamma} = \mu\gamma}$$

- Perturbed Jordan blocks

$$J_\nu = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \nu & & & 0 \end{pmatrix}, \quad \nu \in (0,1)$$

have the form $V \Lambda V^{-1}$ **and** satisfy $(p(J_\nu))^T \Pi$ is complex symmetric. Therefore above equation can be written as

$$X \Sigma X^T \bar{\gamma} = \mu\gamma,$$

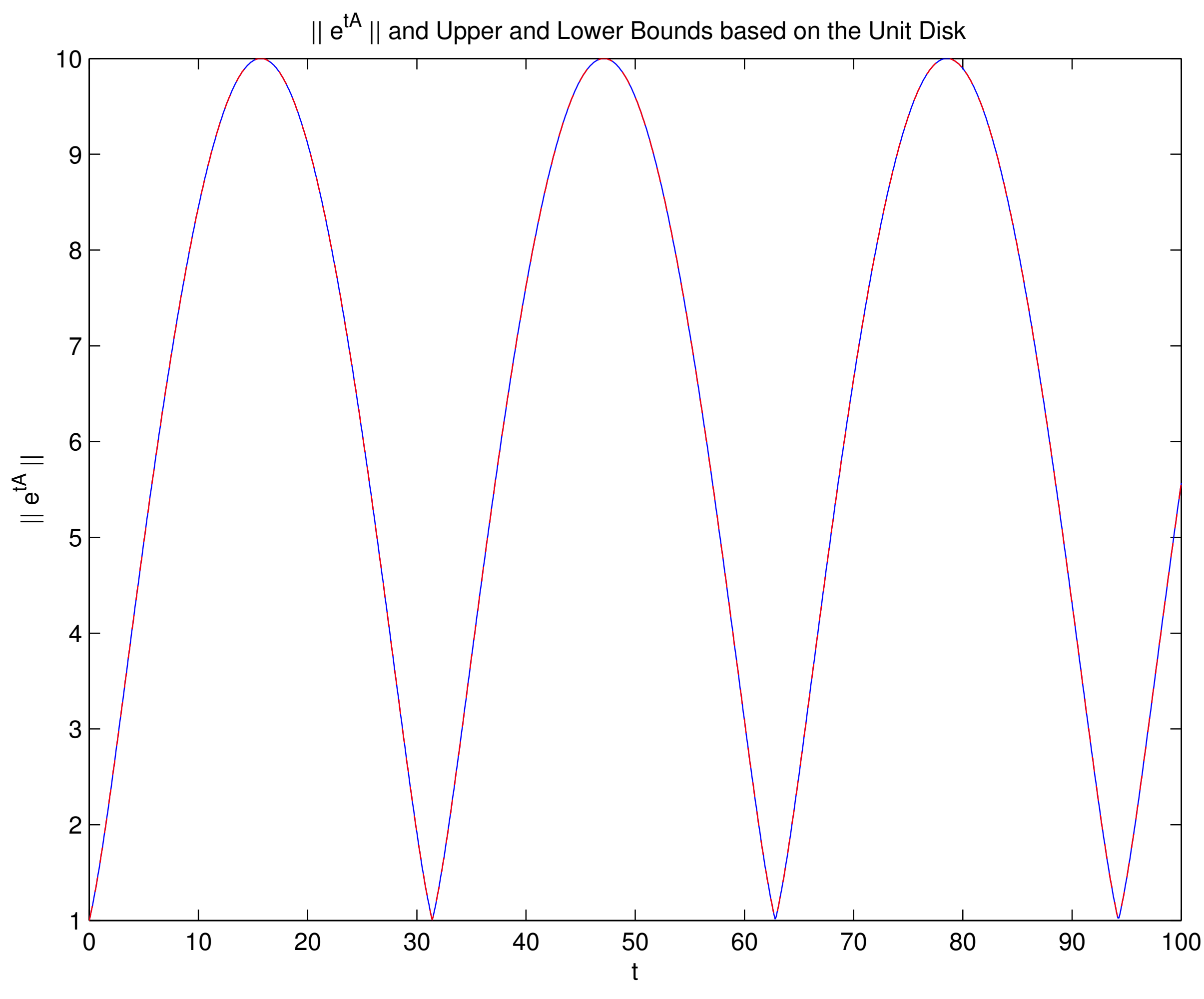
where $X \Sigma X^T$ is SVD of $p(J_\nu)^T \Pi$. Solutions are $\gamma = \mathbf{x_j}, i\mathbf{x_j}$, $\mu = \pm\sigma_j$. Thus $\mu = \sigma_1 = \|p(J_\nu)\|$; i.e., $\forall p$

$$\|p(J_\nu)\| = \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(J_\nu) = p(J_\nu)\},$$

so $m = M = 1$.

Example:

$$A = \begin{pmatrix} 0 & 1 \\ -.01 & 0 \end{pmatrix}$$



Crouzeix's Conjecture: For any matrix A and any polynomial p,

$$\|p(A)\| \leq 2 \max_{z \in W(A)} |p(z)|.$$

Equivalently,

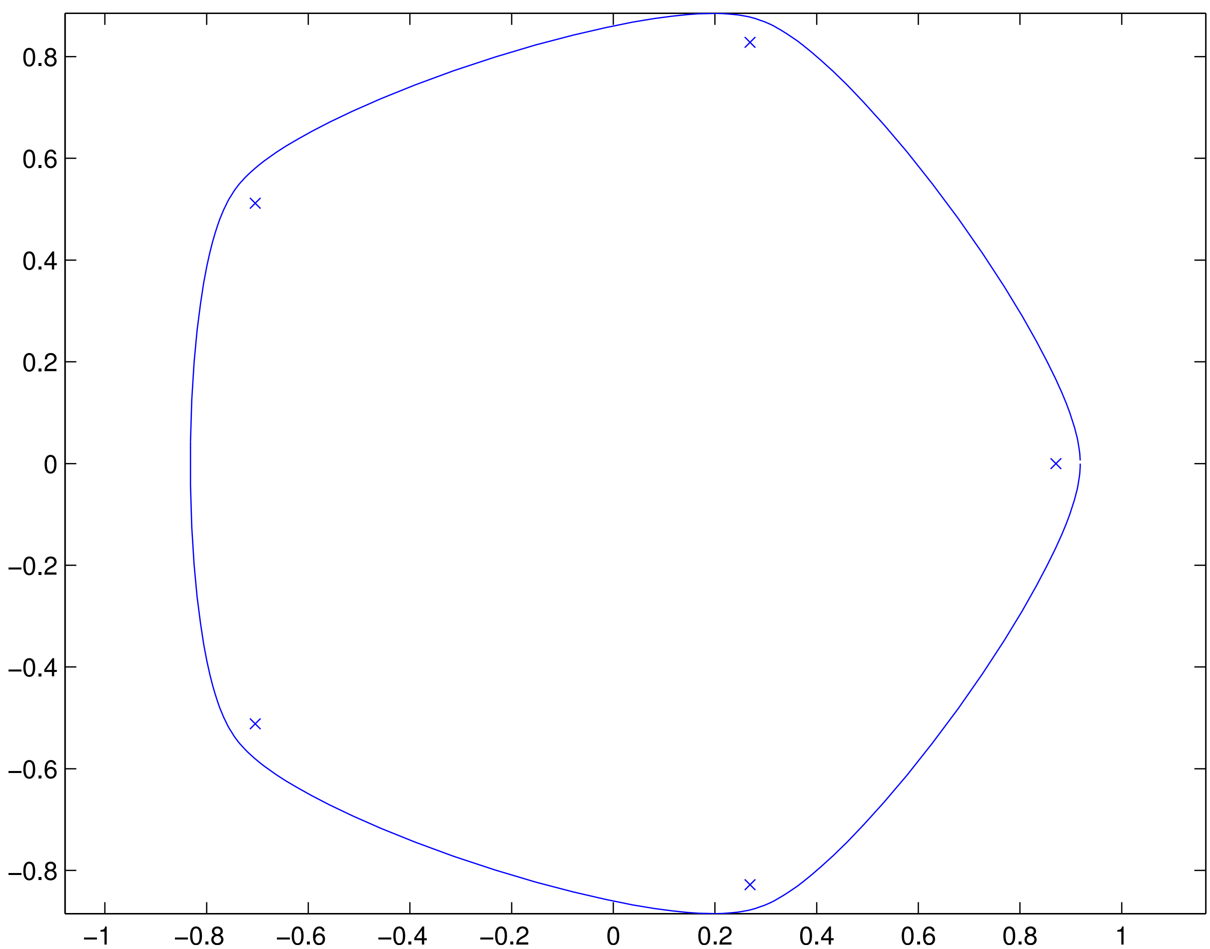
$$\|p(A)\| \leq 2 \inf\{\|f\|_{\mathcal{L}^\infty(W(A))} : f(A) = p(A)\}.$$

Can we prove this for J_ν? Is

$$\|p(J_\nu)\| = \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(J_\nu) = p(J_\nu)\} \leq$$

$$2 \inf\{\|f\|_{\mathcal{L}^\infty(W(J_\nu))} : f(J_\nu) = p(J_\nu)\}?$$

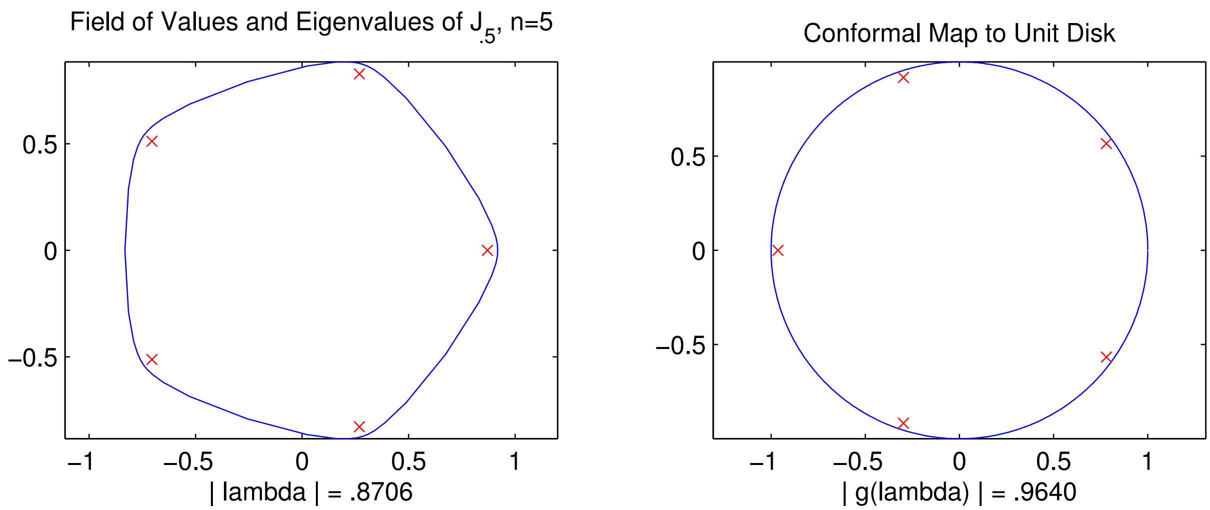
Field of Values and Eigenvalues of $J_{.5}$, n=5



Original eigenvalues are uniformly spaced around circle of radius $\nu^{1/n}$. Map field of values to unit disk. Mapped eigenvalues are uniformly spaced around circle of radius ? Show

$$\inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(\lambda_j) = w_j\} \leq$$

$$2 \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(g(\lambda_j)) = w_j\}.$$



$$W(J_{\nu_{n\times n}}) \supset W(J_{n-1\times n-1}) = \mathcal{D}(0, \cos(\pi/n)).$$

$$\|p(J_\nu)\| \leq ?\inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D}(0,\cos(\pi/n)))} : f(J_\nu) = p(J_\nu)\}$$

$$\|V^{-T}p(\Lambda)V^T\| \leq ?\|D^{-1}V^{-T}p(\Lambda)V^T D\|,$$

$$D = \begin{pmatrix} 1 & & & \\ & \cos(\pi/n) & & \\ & & \ddots & \\ & & & (\cos(\pi/n))^{n-1} \end{pmatrix}.$$

$? = \kappa(D) = (\cos(\pi/n))^{n-1}.$

$\kappa(D) \to 1$ as $n \to \infty$.

$\kappa(D) < 2$ if $n > 6$. For $n = 3$, $\kappa(D) = 4$.