Upper and Lower Bounds on Norms of Functions of Matrices

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Given an n by n matrix A, find a set $S \subset \mathbf{C}$ that can be associated with A to give more information than the spectrum alone can provide about the 2-norm of functions of A.

- Field of values:

$$\mathbf{W}(\mathbf{A}) = \{\langle \mathbf{Aq}, \mathbf{q} \rangle : \ \langle \mathbf{q}, \mathbf{q} \rangle = \mathbf{1}\}.$$

- ϵ-pseudospectrum:

$$\sigma_\epsilon(\mathbf{A}) = \{\mathbf{z} \in \mathbf{C} : \ z \text{ is an eigenvalue of } A + E \text{ for some } E \text{ with } \|E\| < \epsilon\}.$$

- Polynomial numerical hull of degree k:

$$\mathcal{H}_\mathbf{k}(\mathbf{A}) = \{\mathbf{z} \in \mathbf{C} : \ \|\mathbf{p}(\mathbf{A})\| \geq |\mathbf{p}(\mathbf{z})| \ \forall \mathbf{p} \in \mathcal{P}_\mathbf{k}\}.$$

Choose $S \subset \mathbf{C}$. For each p consider:

$$\inf\{\|f\|_{\mathcal{L}^\infty(S)} : \; f(A) = p(A)\}. \qquad (*)$$

Find scalars m and M such that for all p:

$$m \cdot (*) \leq \|p(A)\| \leq M \cdot (*).$$

- $f(A) = p(A)$ if $f(z) = p_{r-1}(z) + \chi(z)h(z)$ for some $h \in H^\infty(S)$. Here χ is the minimal polynomial (of degree r) and p_{r-1} is the polynomial of degree $r-1$ that matches p at the eigenvalues of A and whose derivatives of order up through $t-1$ match those of p at an eigenvalue corresponding to a t by t Jordan block.

- $(*)$ is a **Nevanlinna-Pick interpolation problem**.

Given $S \subset \mathbf{C}$, $\lambda_1, \ldots, \lambda_n \in S$, and $w_1, \ldots, w_n$, find

$$\inf\{\|f\|_{\mathcal{L}^\infty(S)} : \; f(\lambda_j) = w_j, \; j = 1, \ldots, n\}.$$

- If S is the open unit disk, then infimum is achieved by a function $\tilde{f}$ that is a scalar multiple of a finite **Blaschke product**:

$$\tilde{f}(z) = \mu \prod_{k=0}^{n-1} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} = \mu \, \frac{\gamma_0 + \gamma_1 z + \ldots + \gamma_{n-1} z^{n-1}}{\bar{\gamma}_{n-1} + \bar{\gamma}_{n-2} z + \ldots + \bar{\gamma}_0 z^{n-1}}.$$

- Using second representation, Glader and Lindström showed how to compute $\tilde{f}$ and $\|\tilde{f}\|_{\mathcal{L}^\infty(\mathcal{D})}$ by solving a simple eigenvalue problem.

- If S is a simply connected set (such as the field of values) map it conformally to $\mathcal{D}$ ($g : S \to \mathcal{D}$) and solve the problem there:

$$\inf\{\|f\|_{\mathcal{L}^\infty(S)} : f(\lambda_j) = w_j,\ j = 1, \ldots, n\} =$$
$$\inf\{\|F \circ g\|_{\mathcal{L}^\infty(S)} : (F \circ g)(\lambda_j) = w_j,\ j = 1, \ldots, n\} =$$
$$\inf\{\|F\|_{\mathcal{L}^\infty(\mathcal{D})} : F(g(\lambda_j)) = w_j,\ j = 1, \ldots, n\}.$$

- Some results also known for multiply connected sets.

How do we compute the minimal-norm interpolating function $\tilde{f}$?

As noted earlier, it has the form

$$\tilde{f}(z) = \mu \, \frac{\gamma_0 + \gamma_1 z + \ldots + \gamma_{n-1} z^{n-1}}{\bar{\gamma}_{n-1} + \bar{\gamma}_{n-2} z + \ldots + \bar{\gamma}_0 z^{n-1}}.$$

and satisfies $\tilde{f}(\lambda_j) = p(\lambda_j), \; j = 1, \ldots, n.$

Let V be the Vandermonde matrix for $\lambda_1, \ldots, \lambda_n$:

$$V^T = \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{pmatrix}.$$

If $\gamma = (\gamma_0, \ldots, \gamma_{n-1})^T$, and Π is the permutation matrix with 1's on its skew diagonal, then these conditions are:

$$V^{-T} p(\Lambda) V^T \Pi \bar{\gamma} = \mu \gamma.$$

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Glader and Lindström showed that there is a real scalar μ for which this equation has a nonzero solution vector γ and that the largest such μ is $||\tilde{f}||_{\mathcal{L}^{\infty}(\mathcal{D})}$. This is a *coneigenvalue* problem; equate real and imaginary parts to get a $2n$ by $2n$ eigenvalue problem.

$$V^{-T} p(\Lambda) V^T \Pi \bar{\gamma} = \mu \gamma.$$

- **Companion matrices** (with eigenvalues in $\mathcal{D}$) have the form $V \Lambda V^{-1}$, where V is the same Vandermonde matrix as above. Therefore above equation becomes

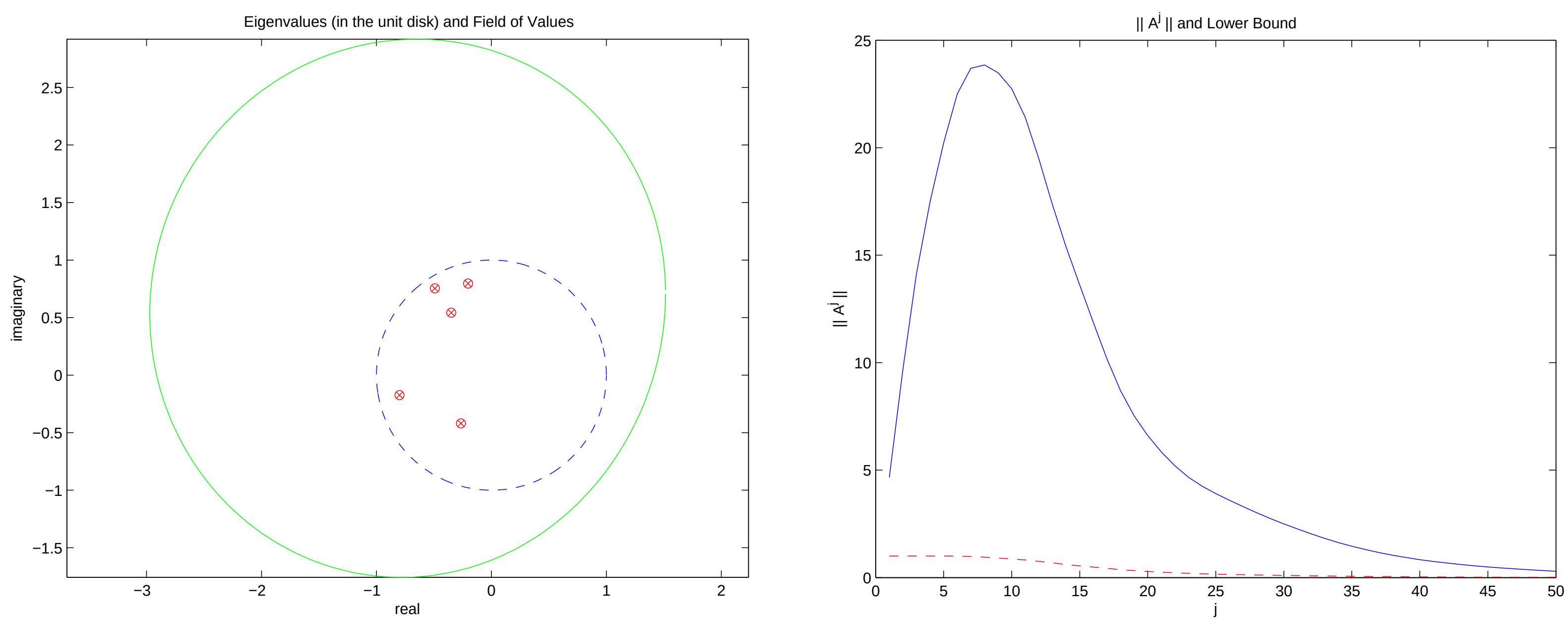
$$(p(A))^T \Pi \bar{\gamma} = \mu \gamma \Rightarrow \|p(A)\| \geq \mu;$$

i.e., $\forall p$

$$\|p(A)\| \geq \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(A) = p(A)\},$$

so $m = 1$. In general, do not have good values for M.

Example: Companion matrix with 5 random eigenvalues in the unit disk. $||A^j||$ and lower bound.



$$V^{-T}p(\Lambda)V^T\Pi\bar{\gamma} = \mu\gamma.$$

- **Perturbed Jordan blocks**

$$J_\nu = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \nu & & & 0 \end{pmatrix}, \quad \nu \in (0,1)$$

have the form $V\Lambda V^{-1}$ **and** $p(J_\nu)^T\Pi$ is *complex symmetric*. Therefore SVD of $p(J_\nu)^T\Pi$ has the form $X\Sigma X^T$, and above equation can be written as

$$X\Sigma X^T\bar{\gamma} = \mu\gamma.$$

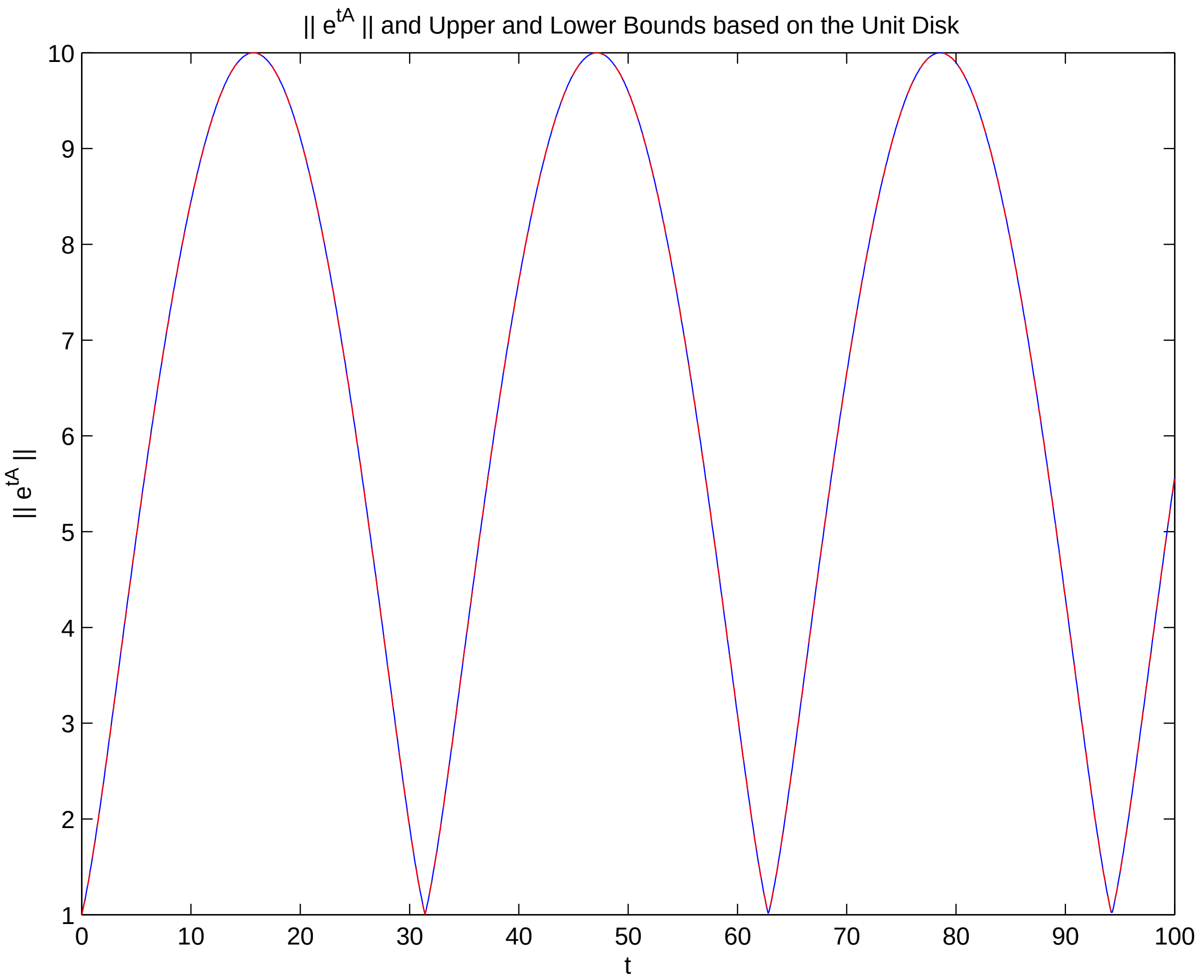
Solutions are $\gamma = \mathbf{x_j},\ i\mathbf{x_j}$, $\mu = \pm\sigma_j$. Thus $\mu = \sigma_1 = \|p(J_\nu)\|$; i.e., $\forall p$

$$\|p(J_\nu)\| = \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(J_\nu) = p(J_\nu)\},$$

so $m = M = 1$.

Example:

$$A = \begin{pmatrix} 0 & 1 \\ -.01 & 0 \end{pmatrix}$$



Crouzeix's Conjecture: For any matrix A and any polynomial p,

$$\|p(A)\| \leq 2 \max_{z \in W(A)} |p(z)|.$$

Equivalently,

$$\|p(A)\| \leq 2 \ \inf\{\|f\|_{\mathcal{L}^\infty(W(A))} : f(A) = p(A)\}.$$

This was proved for $n = 2$ (Crouzeix) or if $W(A)$ is a disk (Badea). For all A, Crouzeix proved it with constant 11.08 instead of 2.

Does Crouzeix's Conjecture hold for J_ν?

- If $\nu \geq 2^{-n/(n-1)}$, then $\kappa(V) \leq 2$. Hence

$$\|p(J_\nu)\| \leq 2 \max_{z \in \sigma(J_\nu)} |p(z)| \leq 2 \inf\{\|f\|_{\mathcal{L}^\infty(W(J_\nu))} : f(J_\nu) = p(J_\nu)\}.$$

- Assume $\nu < 2^{-n/(n-1)}$. $g : W(J_\nu) \to \mathcal{D}$ conformal. Show

$$\|p(J_\nu)\| = \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(\lambda_j) = p(\lambda_j), j = 1, \ldots, n\} \leq$$

$$2 \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(g(\lambda_j)) = p(\lambda_j),\ j = 1, \ldots, n\},$$

where $\lambda_j = \nu^{1/n} e^{2\pi i j/n}$.

- Let $\mathcal{D}_r$ be the disk about the origin of radius r and assume that $\mathcal{D}_r$ contains the eigenvalues of J_ν; i.e., $\nu < r^n$. Then

$$\|p(J_\nu)\| \leq r^{-(n-1)} \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D}_r)} : f(J_\nu) = p(J_\nu)\}.$$

- $W(J_\nu) \supset \mathcal{D}_{\cos\frac{\pi}{n}} \Rightarrow$

$$\|p(J_\nu)\| \leq \left(\cos\frac{\pi}{n}\right)^{-(n-1)} \inf\{\|f\|_{\mathcal{L}^\infty(W(J_\nu))} : f(J_\nu) = p(J_\nu)\}.$$

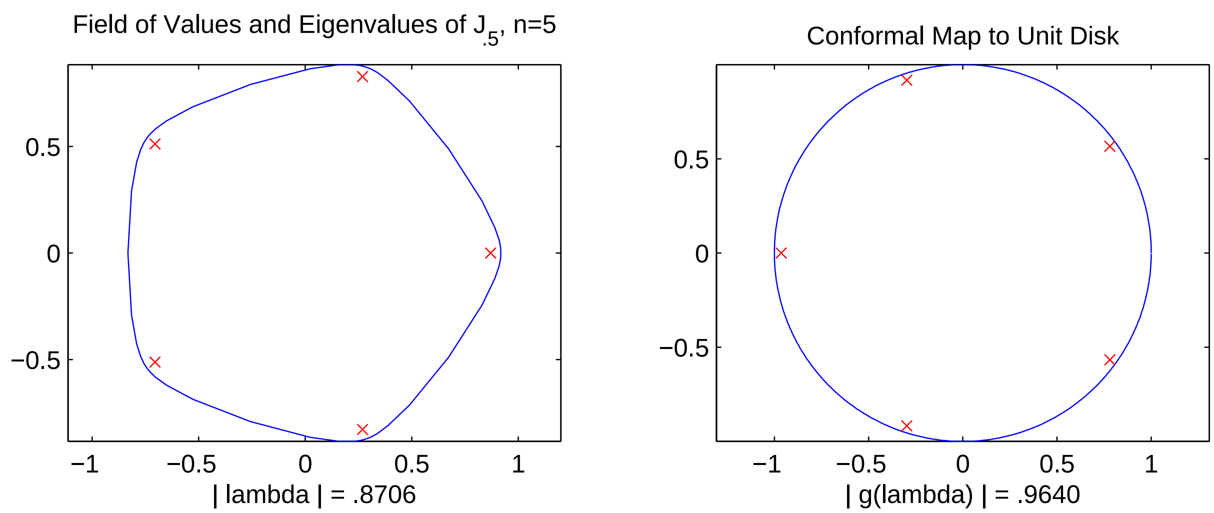
$\cos(\pi/n)^{-(n-1)} \searrow 1$ as $n \nearrow \infty$.

$\cos(\pi/n)^{-(n-1)} < 2$ for $n > 6$, so Crouzeix's conjecture holds in this case.

$\cos(\pi/n)^{-(n-1)} \leq 4$ for $n = 3, 4, 5, 6$, so inequality holds with constant at most 4.

- Can prove conjecture for $n = 6$ by looking at a slightly larger disk inside $W(J_\nu)$.

- For $n = 3, 4, 5$, must look more closely at $W(J_\nu)$ and its map to $\mathcal{D}$. Have not yet established constant 2 in this case.



Numerical Testing of Crouzeix's Conjecture

$$\|p(A)\| \stackrel{?}{\leq} 2 \ \inf\{\|f\|_{\mathcal{L}^\infty(W(A))} : f(A) = p(A)\}$$

- Given A, compute eigendecomposition $A = S\Lambda S^{-1}$, field of values $W(A)$ (or inner and outer polygons), conformal mapping $g : W(A) \to \mathcal{D}$, and $g(\Lambda)$.

- Try values $w_1, \ldots, w_n$ for $p(\lambda_1), \ldots, p(\lambda_n)$. Compute $\|p(A)\| = \|Sp(\Lambda)S^{-1}\|$, and find

$$\mu \equiv \inf\{\|f\|_{\mathcal{L}^\infty(\mathcal{D})} : f(g(\lambda_j)) = w_j, \ j = 1, \ldots, n\}$$

  by solving eigenvalue problem.

- Vary $w_1, \ldots, w_n$ to minimize $\mu/\|p(A)\|$. If $< \frac{1}{2}$, conjecture is false.

  Experiments show that for some problems (e.g. 3×3 perturbed Jordan block with small ν) need (almost) exact $W(A)$ and $g : W(A) \to \mathcal{D}$ to obtain constant 2.