Assignment 1. Due Wednesday, Jan. 19.

Reading: Horn and Johnson, secs. 1.0–1.3, 1.5.

1. Let A be an n by n matrix. Show that the following two statements are equivalent:

   (a) A has a complete set of orthonormal eigenvectors; that is, A can be written in the form $A = U\Lambda U^*$ where U is a unitary matrix and Λ is a diagonal matrix of eigenvalues.

   (b) A commutes with its conjugate transpose; that is, $AA^* = A^*A$, where $(A^*)_{ij} := \overline{A_{ji}}$.

2. (problem 9 on p. 25 in HJ.) Let $J_n(0)$ be the n by n Jordan block with eigenvalue 0:

$$J_n(0) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

   Show that $\mathcal{F}(J_n(0))$ is a disk centered at the origin with radius $\rho(H(J_n(0)))$. Here $\mathcal{F}(\cdot)$ denotes the field of values, $\rho(\cdot)$ the spectral radius, and $H(\cdot)$ the Hermitian part. [Hint: Note that for any θ, $e^{i\theta}J_n(0)$ is unitarily similar to $J_n(0)$ via the unitary similarity transformation $\mathrm{diag}(1, e^{-i\theta}, \ldots, e^{-i(n-1)\theta})J_n(0)\mathrm{diag}(1, e^{i\theta}, \ldots, e^{i(n-1)\theta})$.] Use this to show that $\mathcal{F}(J_n(0))$ is strictly contained in the unit disk. If $D \in \mathbf{C}^{n\times n}$ is diagonal, show that $\mathcal{F}(DJ_n(0))$ is also a disk centered at the origin with radius $\rho(H(DJ_n(0))) = \rho(H(|D|J_n(0)))$.

3. Write a MATLAB code to compute the field of values of a matrix. Use your code to compute the field of values of a 50 by 50 matrix with -1's on the first subdiagonal, 1's on the main diagonal and the first three superdiagonals, and zeros elsewhere. Turn in a plot of your result, using the command `axis equal`, so that you see the actual shape. Also compute the numerical radius: $\nu(A) := \max_{z\in\mathcal{F}(A)} |z|$.