Assignment 3. Due Friday, Apr. 24.

Reading: Through section on Fourier Transforms in the Notes.

1. (1994 prelim, problem 4) Let Δ denote the Laplacian in $\mathbf{R}^n$; i.e.,

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

   We denote by $\mathcal{S}(\mathbf{R}^n)$ the Schwartz space. Let $f \in \mathcal{S}(\mathbf{R}^n)$. Prove that there exists a unique $u \in \mathcal{S}(\mathbf{R}^n)$ such that

$$\Delta u - u = f.$$

2. (2002 prelim, problem 8) Consider the following Cauchy problem for the function $u(x,t) = (u_1(x,t), u_2(x,t))^T$ defined for all $x \in \mathbf{R}$ and for all $t \in \mathbf{R}$:

$$\text{PDE: } u_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_x, \quad \text{Initial Conditions: } u(x,0) = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}.$$

   (a) Take the Fourier transform in x and derive a formula for $\hat{u}(\xi,t)$ in terms of the Fourier transform $\hat{f}(\xi)$ of f.

   (b) Assuming that f is in the Schwartz space $\mathcal{S}(\mathbf{R})$, use your formula from part (a) to find an explicit expression for the solutions $u(x,t)$ involving only f (not its Fourier transform $\hat{f}$).

3. (2002 prelim, problem 1) Let H_+ be the subspace of $L^2(\mathbf{R}^n)$ consisting of all functions $f(x)$ such that $\|f\|_+ < \infty$, where

$$\|f\|_+ := \int_{\mathbf{R}^n} e^{2|\xi|} |\hat{f}(\xi)|^2 \, d\xi;$$

   here $\hat{f}(\xi)$ denotes the Fourier transform of f. Let $a(x)$ be a function in $L^1(\mathbf{R}^n)$ for which $\hat{a}(\xi)e^{|\xi|} \in L^1(\mathbf{R}^n)$.

   Show that the multiplication operator $M : f \to af$ is a bounded linear operator from $(H_+, \|\cdot\|_+)$ to $(H_+, \|\cdot\|_+)$.

4. Do problem 1 on problem set 2 in the Notes.