# Assignment 3. Due Friday, Jan. 30.

Reading: Coddington and Levinson, Ch. 1 (any parts you have not yet read).
Course Notes, through p. 34.

1. Let $n = 1$, $\mathbf{F} = \mathbf{R}$. Let $f(u)$ be a positive continuous function on $[u_0, \infty)$. Consider the IVP: $u' = f(u)$, $u(0) = u_0$.

   **(a)** Use the inverse function theorem to give a rigorous justification of the method of "separation of variables" to solve this problem by proving that the equation
   $$\int_{u_0}^{u} \frac{dv}{f(v)} = t$$
   determines a $C^1$ function $u(t)$ which is the unique solution of the IVP for $t \geq 0$.

   **(b)** Show that the solution to this IVP exists for all time $t \geq 0$ if and only if
   $$\int_{u_0}^{\infty} \frac{dv}{f(v)} = \infty.$$

2. Let $n = 1$, $\mathbf{F} = \mathbf{R}$. Suppose $U : \mathbf{R} \to \mathbf{R}$ is $C^1$. If we interpret $U(x)$ as the potential energy of a particle at position $x$, then $-\frac{d}{dx}U(x)$ is the force acting on the particle. Assuming the particle has mass 1, Newton's equation of motion is $x''(t) = -\frac{d}{dx}U(x(t))$.

   **(a)** Show how to write this equation as a first-order system.

   **(b)** The *kinetic energy* of the particle is $\frac{1}{2}(x')^2$, so the total energy is $E(t) = \frac{1}{2}(x'(t))^2 + U(x(t))$. Show that if $x$ solves Newton's equation then $E(t)$ is constant; i.e., energy is conserved.

   **(c)** Suppose that $U$ is bounded from below; i.e., there exists a constant $C \in \mathbf{R}$ such that $U(x) \geq C$ for all $x \in \mathbf{R}$. Prove that every solution of Newton's equation exists for all time $t \in (-\infty, +\infty)$.

   **(d)** Show that if $U(x) = -x^4$, then the solution of Newton's equation satisfying the initial conditions $x(0) = 0$, $x'(0) = 1$ blows up in finite time.

3. Extend the 'Fundamental Estimate' on p. 26 in the Notes to the case of *piecewise* $C^1$ $\epsilon$-approximate solutions. That is, define $x(t)$ to be an $\epsilon$-approximate solution of the differential equation $x' = f(t, x)$ on some interval $I$ if (1) $x \in C^1$ on $I$, except possibly for a finite set of points $S$ in $I$, where the right and left limits of $x'(t)$ exist but are not equal, and (2) $x(t)$ satisfies $|x'(t) - f(t, x(t))| \leq \epsilon$ for $t \in I \setminus S$. With this definition, prove the following:

   Let $f(t, x)$ be in $(C, Lip)$ on a domain $D$, with Lipschitz constant $L$. Suppose $x_1(t)$ is an $\epsilon_1$-approximate solution and $x_2(t)$ is an $\epsilon_2$-approximate solution of the differential equation $x' = f(t, x)$ on some interval $I$ with $t_0 \in I$ and $\{(t, x(t)) : t \in I\} \subset D$. Assume also that $\{(t, x_i(t)) : t \in I\} \subset D$, $i = 1, 2$, and that $|x_1(t_0) - x_2(t_0)| \leq \delta$. Then for $t \in I$,

$$|x_1(t) - x_2(t)| \leq \delta e^{L|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{L} \left( e^{L|t-t_0|} - 1 \right).$$

4. (1994 prelim, problem 6) Fix $T > 0$. Let $a_0 : [0, T] \to \mathbf{R}$ be continuous, $a_0(t) \geq 0$ for all $t \in [0, T]$. Also assume that $a_1(t)$ is continuous. Assume there exists a continuously differentiable $x : [0, T] \to \mathbf{R}$ satisfying the ODE:

$$x'(t) = a_0(t) \cdot (x(t))^2 + a_1(t) \cdot x(t), \quad 0 < t < T.$$

Show that if $x(0) > 0$, then

$$\int_0^T \left( a_0(t) \cdot e^{-\int_0^t a_1(s)\, ds} \right) dt < (x(0))^{-1}.$$

Hint: Consider first the case $a_1(t) = 0$, $0 \leq t \leq T$.

5. (1996 prelim, problem 8) For any $A \in \mathbf{C}^{n \times n}$, let

$$\mu_h(A) = \frac{\|I + hA\| - 1}{h} \quad \text{for } h > 0,$$

where $\|\cdot\|$ is the matrix 2-norm.

**(a)** Show that $\mu_h(A)$ decreases monotonically as $h \searrow 0$. Hint: You may use the fact that the 2-norm is strictly convex, so $f(h) \equiv \|I + hA\|$ is a convex function.

**(b)** Show that $\mu_h(A) \geq -\|A\|$ for all $h > 0$.

From (a) and (b) we can conclude that

$$\mu(A) \equiv \lim_{h \searrow 0} \mu_h(A)$$

exists and is finite for all $A \in \mathbf{C}^{n \times n}$.

$\mu(A)$ is called the *logarithmic norm* of $A$.

**(c)** Let $y(t)$ satisfy $y'(t) = A(t)y(t)$ with $y(0) = y_0 \in \mathbf{C}^n$, where $A(t) : \mathbf{R}^+ \to \mathbf{C}^{n \times n}$ is continuously differentiable. Show that

$$\frac{d}{dt}\|y(t)\| \leq \mu(A(t)) \cdot \|y(t)\|.$$

Hence, in particular, if $\mu(A(t)) \leq 0$ for all $t \geq 0$, then solutions to the initial value problem remain bounded,

$$\|y(t)\| \leq \|y_0\| \quad \text{for all } t \geq 0.$$

Hint: Use a Taylor expansion of $y$.