Assignment 2. Due Friday, Jan. 25.

Reading: Course Notes, through p. 19.
Coddington and Levinson, Ch. 1, secs. 7–8.

The first two problems are uniqueness theorems with weaker hypotheses than the Lipschitz condition.

1. One-sided uniqueness theorem ($n = 1$, $\mathbf{F} = \mathbf{R}$)

   (a) A real-valued function $f(t, u)$ is said to satisfy a *one-sided* Lipschitz condition in u if there is a constant L such that $(\forall u_1, u_2, t \in \mathbf{R})$

   $$u_2 > u_1 \Rightarrow f(t, u_2) - f(t, u_1) \leq L(u_2 - u_1).$$

   Show that if f is continuous in t and u and satisfies a one-sided Lipschitz condition in u, then there is at most one solution of the IVP $u' = f(t, u)$, $u(t_0) = 0$, for $t \geq t_0$.

   (b) Let $f(t, u)$ be real-valued, continuous in t and u, and decreasing (not necessarily strictly) in u for each t; i.e., $u_2 > u_1 \Rightarrow f(t, u_2) \leq f(t, u_1)$. Show that if $u(t)$ and $v(t)$ are both solutions of $u' = f(t, u)$, then $|u(t) - v(t)| \leq |u(s) - v(s)|$ for $t \geq s$. Deduce uniqueness for the IVP $u' = f(t, u)$, $u(t_0) = 0$ for $t \geq t_0$. Show, however, that uniqueness may fail for $t < t_0$.

2. Let $f(t, x)$ be continuous of $[0, a] \times \mathbf{R}^n$ (mapping into $\mathbf{R}^n$) and satisfy the *generalized Lipschitz condition*

   $$|f(t, x) - f(t, y)| \leq \kappa(t)|x - y| \quad (\forall t \in [0, a]) \quad (\forall x, y \in \mathbf{R}^n),$$

   where $\kappa(t) \geq 0$ and κ is continuous on $(0, a]$ but possibly unbounded near $t = 0$. Show that if $\int_0^a \kappa(t)\, dt < \infty$, then the IVP $x' = f(t, x)$, $x(0) = x_0$, has at most one solution on $[0, a]$.

3. *Integral Forms of Gronwall's Inequality:* Let φ, ψ, and α be real-valued continuous functions on the interval $I = [a, b]$. Suppose $\alpha \geq 0$ on I and

   $$\varphi(t) \leq \psi(t) + \int_a^t \alpha(s)\varphi(s)\, ds \quad (\forall t \in I).$$

   (a) Show that for each $t \in I$,

   $$\varphi(t) \leq \psi(t) + \int_a^t \exp\left(\int_s^t \alpha(r)\, dr\right) \alpha(s)\psi(s)\, ds.$$

   (Hint: Let $u(t) = \int_a^t \alpha(s)\varphi(s)\, ds$ and show that $u' - \alpha u \leq \alpha\psi$.)

(b) Suppose $\psi(t) \equiv c$ is constant. Show that for each $t \in I$, $\varphi(t) \leq c \exp\left(\int_a^t \alpha(s)\, ds\right)$.

4. Let $n = 1$, $\mathbf{F} = \mathbf{R}$. Suppose $f(t, u)$ satisfies a Lipschitz condition for $t \geq t_0$. Suppose that $u(t)$ satisfies the differential inequality $u' \leq f(t, u)$ for $t \geq t_0$ and $v(t)$ satisfies $v' = f(t, v)$ for $t \geq t_0$. Suppose $u(t_0) < v(t_0)$. Prove that $u(t) < v(t)$ for $t \geq t_0$.

5. Show that if there are two distinct solutions of $u' = f(t, u)$ ($n = 1$, $\mathbf{F} = \mathbf{R}$) satisfying the same initial condition at t_0, then there are infinitely many.

6. (2001 prelim, problem 4) Consider the equation of harmonic motion:

$$u'' = -ku, \quad u(t_0) = u_0, \quad u'(t_0) = v_0.$$

Here $u(t)$ represents the distance from equilibrium and $k > 0$ is a spring constant.

(a) Write this as a system of two first-order differential equations, and show that the right-hand side of your system satisfies a Lipschitz condition on $\mathbf{R}^2$. Determine the (smallest possible) Lipschitz constant for the 2-norm.

(b) Use Gronwall's inequality to derive a bound on the 2-norm of the difference between $u(t)$ and $\tilde{u}(t)$, $t \geq t_0$, where $\tilde{u}(t)$ satisfies the differential equation with initial conditions $\tilde{u}(t_0) = u_0 + \epsilon$, $\tilde{u}'(t_0) = v_0 + \delta$.