Assignment 1. Due Friday, Jan. 16.

Reading: Course Notes, through p. 19.
Coddington and Levinson, Ch. 1, secs. 1–6.
Skim through: Birkhoff and Rota, Ch. 1, secs. 1–6, 9–10; Ch. 2, secs. 1–2, 4; Ch. 3, secs. 1–5, Ch. 6, secs. 1–3.

1. (a) Let $f : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}$ be continuous. Suppose $x : \mathbb{R} \to \mathbb{C}$ is a solution of the $n$th-order equation
\[ x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \tag{1} \]
i.e., for each $t \in \mathbb{R}$, $x^{(n)}(t)$ exists and $x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t))$. Show that $x \in C^n(\mathbb{R})$.
(b) Define $F : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ by $F(t, y) = [y_2, \ldots, y_n, f(t, y_1, \ldots, y_n)]^T$ (so $F$ is continuous). Suppose $y : \mathbb{R} \to \mathbb{C}^n$ is a solution of the first-order system
\[ y' = F(t, y); \tag{2} \]
i.e., for each $t \in \mathbb{R}$, $y'(t)$ exists and $y'(t) = F(t, y(t))$. Show that $y \in C^1(\mathbb{R})$, and moreover for $1 \leq j \leq n$, $y_j \in C^{n-j+1}(\mathbb{R})$.
(c) Show that if $x \in C^n(\mathbb{R})$ is a solution of (1), then $y = [x, x', \ldots, x^{(n-1)}]^T$ is a $C^1$ solution of (2). Moreover, if $x$ satisfies the initial conditions $x^{(k)}(t_0) = x_0^k (0 \leq k \leq n - 1)$, then $y$ satisfies the initial conditions $y(t_0) = [x_0^0, \ldots, x_0^{n-1}]^T$.
(d) Show that if $y$ is a $C^1$ solution of (2), then $x = y_1$ is a $C^n$ solution of (1). Moreover, if $y$ satisfies the initial conditions $y(t_0) = y_0$, then $x$ satisfies the initial conditions $x^{(k)}(t_0) = (y_0)^{k+1} (0 \leq k \leq n - 1)$.
(e) Show that the first-order system corresponding to the linear $n$th-order equation $x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = b(t)$ is of the form $y' = A(t)y + B(t)$ where $A(t) \in \mathbb{C}^{n \times n}$ and $B(t) \in \mathbb{C}^n$, and identify $A(t)$ and $B(t)$.

2. For each of the following IVP’s, compute the Picard iterates and identify the solution to which they converge.
(a) $x' = tx, \quad x(0) = 1. \ (x$ is a scalar function of $t.)$
(b) $x' = Ax, \quad x(0) = x_0. \ (A \in \mathbb{C}^{n \times n}$ is a constant matrix and $x : \mathbb{R} \to \mathbb{C}^n$.)

3. Let $f \in C \ (n = 1)$ on the rectangle $0 \leq t \leq a, \ |x| \leq b$, where $a, b > 0$, and assume that $f(t, x_1) \leq f(t, x_2)$ if $x_1 \leq x_2$, and $f(t, 0) \geq 0$ for $0 \leq t \leq a$. Prove that the successive approximations in the Picard iteration:
\[ x_{k+1}(t) = \int_0^t f(s, x_k(s)) \, ds, \quad k = 0, 1, 2, \ldots \]
converge to a solution of $x' = f(t, x), \ x(0) = 0$, on $[0, \min\{a, b/M\}]$, where $M = \max |f|$ on the rectangle.
4. For each pair of positive constants $M$ and $T$, define a norm $\| \cdot \|_{M,T}$ on $C([0, T])$ by
\[
\|x\|_{M,T} = \sup_{0 \leq t \leq T} |e^{-tM} x(t)|.
\]

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz continuous and that $T > 0$ is given. Show that the mapping
\[
\Phi : x \to x_0 + \int_0^t f(x(s)) ds
\]
is a contraction on $C([0, T])$ in the $\| \cdot \|_{M,T}$ norm as long as $M$ is large enough.

(b) Apply this to show the existence of a unique solution $x \in C^1([0, T])$ to the initial value problem
\[
x' = f(x), \quad x(0) = x_0
\]
on a finite interval $[0, T]$ of any length $T > 0$.

5. Consider the initial-value problem
\[
\begin{align*}
y'(t) &= f(y(t/2)) \text{ for } t \geq 0, \\
y(0) &= y_0,
\end{align*}
\]where $f : \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz continuous with Lipschitz constant $L$ and $y_0$ is a given constant. This is a “differential delay” equation: notice that $y$ on the right-hand side above is evaluated at $t/2$, not at $t$, so this is not a standard ODE. Prove that there exists a unique solution of this “differential delay” initial-value problem in $C^1[0, \infty)$. 