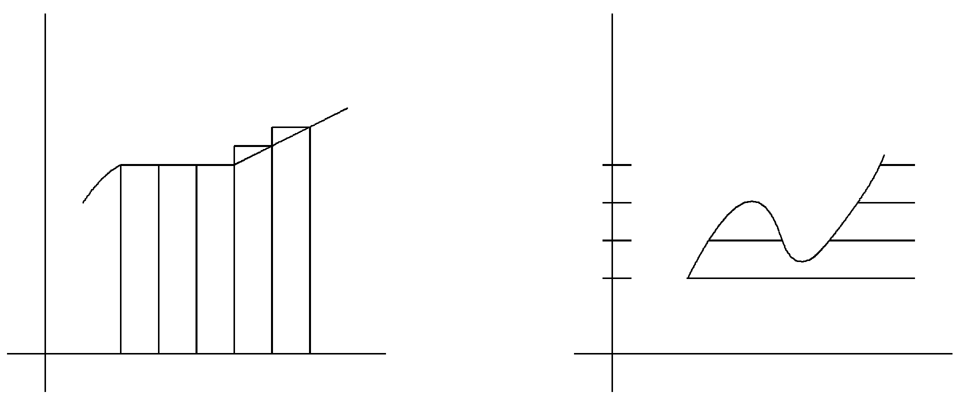
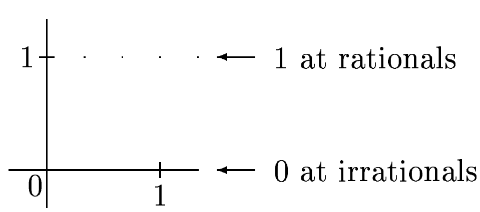
Overview of Lebesgue Integration on $\mathbb{R}^n$

(see Jones, Chapter 2 ff)

Riemann integration is based on subdividing the *domain* of f. The problem with Riemann integration is the requirement of some "smoothness" of f: for x, y close, $f(x)$ and $f(y)$ need to have something to do with each other. Lebesgue integration is based on subdividing the *codomain* of f: it is built on inverse images.



Typical Example. For a set $E \subset \mathbb{R}^n$, define the characteristic function of the set E to be $\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$. Consider $\int_0^1 \chi_{\mathbb{Q}}(x)dx$ ($\mathbb{Q}$ is set of rationals)



Riemann Upper integral - inf of "upper sums" $\overline{\int_0^1} \chi_{\mathbb{Q}}(x)dx = 1$

Lower integral - sup of "lower sums" $\underline{\int_0^1} \chi_{\mathbb{Q}}(x)dx = 0$

Since $\overline{\int_0^1} \chi_{\mathbb{Q}}(x)dx \neq \underline{\int_0^1} \chi_{\mathbb{Q}}(x)dx$, *not* Riemann integrable.

Lebesgue integral: $\lambda(E)$ denotes "size" of E (temporary)

$$\begin{aligned}\int_0^1 \chi_{\mathbb{Q}}(x)dx &= 1 \cdot \lambda(\mathbb{Q} \cap [0,1]) + 0 \cdot \lambda(\mathbb{Q}^c \cap [0,1]) \\ &= 1 \cdot 0 + 0 \cdot 1 = 0.\end{aligned}$$

First, must develop theory of Lebesgue measure (for "size" of sets).

Advantages of Lebesgue theory over Riemann theory:

1. Extension to more functions (on finite intervals).

2. Good convergence theorems: $\lim_{n\to\infty} \int f_n(x)dx = \int \lim_{n\to\infty} f_n(x)dx$ under mild assumptions.

3. Completeness of L^p spaces.

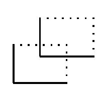
Current goal. Construct Lebesgue measure on $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$, we want to define $\lambda(A)$, the Lebesgue measure of A, with $0 \le \lambda(A) \le \infty$ (an extension of n-dim volume).

Problem: We can't define $\lambda(A)$ for *all* subsets $A \subset \mathbb{R}^n$ and maintain all the properties we want. We will define $\lambda(A)$ for "[Lebesgue] measurable" subsets of $\mathbb{R}^n$ (very many subsets). Construct $\lambda(A)$ for increasingly complicated sets.

Step 0 ϕ $\quad \lambda(\phi) = 0$

Step 1 "Special rectangles" $I = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$, $\lambda(I) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$ for $-\infty < a_j < b_j < \infty$ (note: Jones leaves right ends closed).

Step 2 "Special polygons" P is a finite union of special rectangles, each of finite positive measure.



Fact Every special polygon is a disjoint union of finitely many special rectangles.

For $P = \bigcup_{k=1}^{N} I_k$ (where the I_k's are disjoint, i.e., for $j \neq k$, $I_j \cap I_k = \phi$), define $\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$

Fact $\lambda(P)$ independent of the choice of I_k's.

Step 3 Nonempty open sets $G \subset \mathbb{R}^n$. Define $\lambda(G) = \sup\{\lambda(P) : P \text{ is a special polygon}, P \subset G\}$ ("special poly. from inside").

Remark Every nonempty open set in $\mathbb{R}^n$ can be written as a *countable* disjoint union of special rectangles.

Step 4 Compact sets $K \subset \mathbb{R}^n$. Define $\lambda(K) = \inf\{\lambda(G) : G \text{ open}, K \subset G\}$ ("open sets from outside").

Fact If $K = \overline{P}$ for a special polygon P, then $\lambda(K) = \lambda(P)$.

For $A \subset \mathbb{R}^n$, A arbitrary, define

$$\begin{aligned} \lambda^*(A) &= \inf\{\lambda(G) : G \text{ open}, A \subset G\} \qquad (\textit{outer} \text{ measure of } A) \\ \lambda_*(A) &= \sup\{\lambda(K) : K \text{ compact}, K \subset A\} \qquad (\textit{inner} \text{ measure of } A \end{aligned}$$

Facts. If A is open or compact, $\lambda_*(A) = \lambda(A) = \lambda^*(A)$. Hence for any A, $\lambda_*(A) \leq \lambda^*(A)$.

Step 5 Bounded subsets of $\mathbb{R}^n$: we say a bounded subset $A \subset \mathbb{R}^n$ is [Lebesgue] measurable if $\lambda_*(A) = \lambda^*(A)$, in which case we define $\lambda(A) = \lambda_*(A) = \lambda^*(A)$.

Step 6 Arbitrary subsets of $\mathbb{R}^n$: we say A is [Lebesgue] measurable if for each $R > 0$, $A \cap B(0, R)$ (ball of radius R, center 0) is measurable, and define $\lambda(A) = \sup_{R>0} \lambda(A \cap B(0, R))$.

Let $\mathcal{L}$ denote the collection of all Lebesgue measurable subsets of $\mathbb{R}^n$.

Fact. $\mathcal{L}$ is a σ-algebra of subsets of $\mathbb{R}^n$, i.e., (i) ϕ, $\mathbb{R}^n \in \mathcal{L}$, (ii) $A \in \mathcal{L} \Rightarrow A^C \in \mathcal{L}$, (iii) if $A_1, A_2, \ldots \in \mathcal{L}$ is a *countable* collection of subsets of $\mathbb{R}^n$ in $\mathcal{L}$, then $\bigcup\limits_{k=1}^{\infty} A_k \in \mathcal{L}$.

Fact. If $\mathcal{S}$ is any collection of subsets of a set X, then there is a smallest σ-algebra $\mathcal{A}$ of subsets of X containing $\mathcal{S}$ (i.e., $\mathcal{S} \subset \mathcal{A}$), namely, the intersection of all σ-algebras of subsets of X containing $\mathcal{A}$. This smallest σ-algebra $\mathcal{A}$ is called the σ-algebra *generated* by $\mathcal{S}$.

Definition. The smallest σ-algebra of subsets of $\mathbb{R}^n$ containing the open sets is called the collection $\mathcal{B}$ of *Borel sets.* Closed sets are Borel sets.

Fact. Every open set is [Lebesgue] measurable. Thus $\mathcal{B} \subset \mathcal{L}$.

Fact. If $A \in \mathcal{L}$, then $\lambda_*(A) = \lambda(A) = \lambda^*(A)$.

Caution: However, $\lambda_*(A) = \lambda^*(A) = \infty$ does *not* imply $A \in \mathcal{L}$.

Properties of Lebesgue measure λ

λ is a *measure*: $\lambda(\phi) = 0$, $(\forall\, A \in \mathcal{L})$ $\lambda(A) \geq 0$, and if $A_1, A_2, \ldots \in \mathcal{L}$ are *disjoint* then $\lambda\left(\bigcup\limits_{k=1}^{\infty} A_k\right) = \sum\limits_{k=1}^{\infty} \lambda(A_k)$ (*countable additivity*). Consequences:

(i) if $A, B \in \mathcal{L}$ and $A \subset B$, then $\lambda(A) \leq \lambda(B)$

(ii) if $A_1, A_2, \cdots \in \mathcal{L}$,then $\lambda\left(\bigcup\limits_{k=1}^{\infty} A_k\right) \leq \sum\limits_{k=1}^{\infty} \lambda(A_k)$ (*countable subadditivity*)

Remark: both (i) and (ii) are true of outer measure λ^* on *all* subsets of $\mathbb{R}^n$.

Sets of Measure Zero

Fact. If $\lambda^*(A) = 0$, then $0 \le \lambda_*(A) \le \lambda^*(A) = 0$, so $0 = \lambda_*(A) = \lambda^*(A)$, so $A \in \mathcal{L}$. Thus every subset of a set of measure zero is also measurable (we say λ is a *compact measure*).

Characterization of Lebesgue measurable sets

Definition. A set is called a G_δ if it is the intersection of a countable collection of open sets. A set is called an F_σ if it is the union of a countable collection of closed sets. G_δ sets and F_σ sets are Borel sets.

Fact. A set $A \subset \mathbb{R}^n$ is Lebesgue measurable iff $\exists$ a G_δ set G and an F_σ set F for which $F \subset A \subset G$ and $\lambda(G \backslash F) = 0$. (Note: $G \backslash F = G \cap F^C$ is a Borel set.)

Examples.

(0) If $A = \{a\}$ is a single point, then $A \in \mathcal{L}$ and $\lambda(A) = 0$.

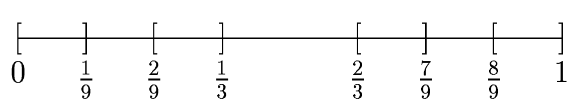
(1) If $A = \{a_1, a_2, \ldots\}$ is countable, then A is measurable, and $\lambda(A) \le \sum_{j=1}^{\infty} \lambda(\{a_j\}) = 0$, so $\lambda(A) = 0$. For example, $\lambda(\mathbb{Q}) = 0$.

(2) $\lambda(\mathbb{R}^n) = \infty$.

(3) **Open sets in $\mathbb{R}$.** Every nonempty open set G in $\mathbb{R}$ is a (finite or) countable *disjoint* union of open intervals (a_j, b_j) $(1 \le j \le J$ or $1 \le j < \infty)$, and $\lambda(G) = \sum_j \lambda(a_j, b_j) = \sum_j (b_j - a_j)$.

(4) **The Cantor Set** is a closed subset of $[0, 1]$. Let

$$
\begin{aligned}
G_1 &= \left(\frac{1}{3}, \frac{2}{3}\right), & \lambda(G_1) = \frac{1}{3} \\
G_2 &= \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right), & \lambda(G_2) = \frac{2}{9} \\
G_3 &= \left(\frac{1}{27}, \frac{2}{27}\right) \cup \cdots \cup \left(\frac{25}{27}, \frac{26}{27}\right), & \lambda(G_3) = \frac{4}{27} \\
\text{etc.} & \quad \left(\text{note } \lambda(G_k) = \frac{2^{k-1}}{3^k}\right)
\end{aligned}
$$



(middle thirds of remaining subintervals)

Let $G = \bigcup_{k=1}^{\infty} G_k$ (G open subset of $(0, 1)$).

Define the Cantor set $C = [0,1]\backslash G$

$$\lambda(G) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = \frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}} = 1$$

Since $\lambda(C) + \lambda(G) = \lambda([0,1]) = 1$, $\lambda(C) = 0$.

Fact. For $x \in [0,1]$, $x \in C$ iff x has a base 3 expansion with only 0's and 2's, i.e., $x = \sum_{j=1}^{\infty} \frac{d_j}{3^j}$ with each $d_j \in \{0,2\}$.

examples.

$$\begin{aligned} 0 &= (0.000\cdots)_3 \\ \tfrac{1}{3} &= (0.100\cdots)_3 = (0.0222\cdots)_3 \\ \tfrac{2}{3} &= (0.200\cdots)_3 \\ 1 &= (0.222\cdots)_3 \end{aligned}$$

$\frac{3}{4} = (0.202020\cdots)_3$ is in C (not an endpoint of any interval in any G_k). C can be put in $1-1$ correspondence with $[0,1]$ (and thus also with $\mathbb{R}$).

Invariance of Lebesgue measure

(1) **Translation**. For a fixed $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, define $x + A = \{x + y : y \in A\}$.

Fact. If $x \in \mathbb{R}^n$ and $A \in \mathcal{L}$, then $x + A \in \mathcal{L}$, and $\lambda(x + A) = \lambda(A)$.

(2) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and $A \in \mathcal{L}$, then $T[A] \in \mathcal{L}$, and $\lambda(T[A]) = |\det T| \cdot \lambda(A)$.

Measurable Functions

Consider a function $f : \mathbb{R}^n \to [-\infty,\infty]$ (arithmetic in the extended reals: for $x \in \mathbb{R}$, $x \pm \infty = \pm\infty$; for $a > 0$, $a \cdot (\pm\infty) = \pm\infty$; for $a < 0$, $a \cdot (\pm\infty) = \mp\infty$; $0 \cdot (\pm\infty) = 0$). f is called *Lebesgue measurable* if for every $t \in \mathbb{R}$, $f^{-1}([-\infty,t]) \in \mathcal{L}$ (in $\mathbb{R}^n$).

Recall. Inverse images commute with unions, intersections, complements

$$f^{-1}[B^C] = f^{-1}[B]^C;\ f^{-1}\left[\bigcup_\alpha A_\alpha\right] = \bigcup_\alpha f^{-1}[A_\alpha];\ f^{-1}\left[\bigcap_\alpha A_\alpha\right] = \bigcap_\alpha f^{-1}[A_\alpha].$$

Fact. For any function $f : \mathbb{R}^n \to [-\infty,\infty]$, the collection of sets $B \subset [-\infty,\infty]$ for which $f^{-1}[B] \in \mathcal{L}$ is itself a σ-algebra of subsets of $[-\infty,\infty]$.

Note. The smallest σ-algebra of subsets of $[-\infty,\infty]$ containing all sets of the form $[-\infty,t]$ for $t \in \mathbb{R}$ contains also $\{-\infty\}$, $\{\infty\}$, and all sets of the form $[-\infty,t)$, $[t,\infty]$, $(t,\infty]$, (a,b), etc. It is the collection of all sets of the form B, $B \cup \{\infty\}$, $B \cup \{-\infty\}$, or $B \cup \{-\infty,\infty\}$ for Borel subsets B of $\mathbb{R}$.

Comments. If f and $g : \mathbb{R}^n \to [-\infty, \infty]$ are measurable, then $f+g$, $f \cdot g$, and $|f|$ are measurable.

(define $\infty + (-\infty)$ (and $(-\infty) + \infty$) separately in a consistent fashion)

Moreover, if $\{f_k\}$ is a sequence of measurable functions $f_k : \mathbb{R}^n \to [-\infty, \infty]$, then so are $\sup_k f_k(x)$, $\inf_k f_k(x)$, $\underbrace{\limsup_k f_k(x)}_{=\inf_{k \geq 1} \sup_{j \geq k} f_j(x)}$, $\liminf_k f_k(x)$, so if $\lim_{k \to \infty} f_k(x)$ exists $\forall\, x$, it is also measurable.

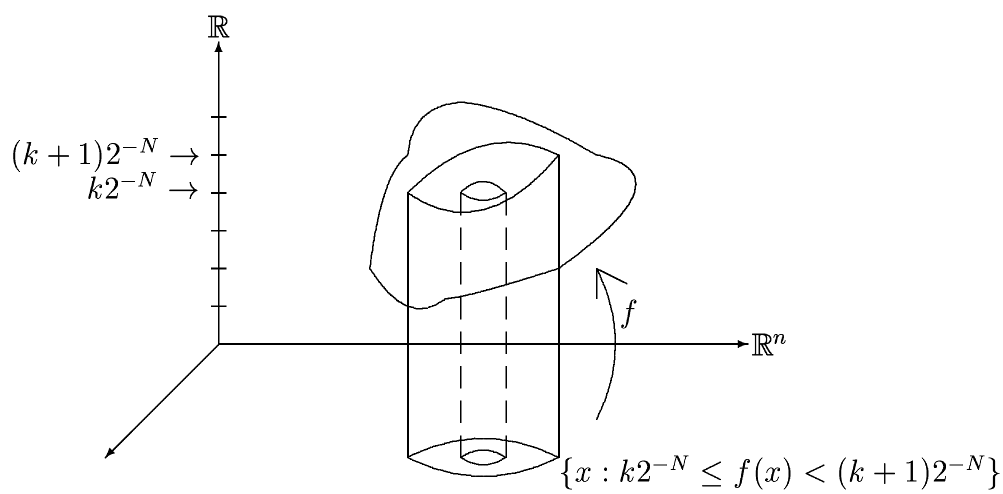
Definition. If $A \subset \mathbb{R}^n$, $A \in \mathcal{L}$, and $f : A \to [-\infty, \infty]$, we say that f is measurable (on A) if, when we extend f to be 0 on A^C, f is measurable on $\mathbb{R}^n$ (equivalent: $f\chi_A$ is measurable for any extension of f).

Definition. If $f : \mathbb{R}^n \to \mathbb{C}$ (*not* including ∞), we say f is Lebesgue measurable if $\mathcal{R}ef$ and $\mathcal{I}mf$ are both measurable.

Fact. $f : \mathbb{R}^n \to \mathbb{C}$ is measurable iff for every open set $G \subset \mathbb{C}$, $f^{-1}[G] \in \mathcal{L}$.

Integration

First consider only $f : \mathbb{R}^n \to [0, \infty]$ (so $f \geq 0$), with f measurable. Define $S_N = \sum_{k=0}^{\infty} k2^{-N} \lambda\left(\{x : k2^{-N} \leq f(x) < (k+1)2^{-N}\}\right) + \underbrace{\infty \cdot \lambda(\{x : f(x) = +\infty\})}_{(\text{here } \infty \cdot 0 = 0)}$. Call this a Lebesgue sum.



Claim. $S_N \leq S_{N+1}$.

Proof. $\{x : k2^{-N} \leq f(x) < (k+1)2^{-N}\}$ is the disjoint union of $\left\{x : k2^{-N} \leq f(x) < \left(k + \frac{1}{2}\right) 2^{-N}\right\}$ and $\left\{x : \left(k + \frac{1}{2}\right) 2^{-N} \leq f(x) < (k+1)2^{-N}\right\}$ so $k2^{-N}\lambda(\{x : k2^{-N} \leq f(x) < (k+1)2^{-N}\}) \leq$

$k2^{-N}\lambda\left(\left\{x : k2^{-N} \leq f(x) < \left(k+\frac{1}{2}\right)2^{-N}\right\}\right)+\left(k+\frac{1}{2}\right)2^{-N}\lambda\left(\left\{x : \left(k+\frac{1}{2}\right)2^{-N} \leq f(x) < (k+1)2^{-N}\right\}\right)$ and the claim follows after summing and redefining indices. $\square$

Define. $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(x)dx = \lim_{N\to\infty} S_N$ (also written $\int_{\mathbb{R}^n} f d\lambda$). This limit exists (in $[0,\infty]$) by the monotonicity $S_N \leq S_{N+1}$.

General Measurable Functions

Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be measurable. Define $f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$ and $f_-(x) = \begin{cases} 0 & \text{if } f(x) > 0 \\ -f(x) & \text{if } f(x) \leq 0 \end{cases}$ so $(\forall x)\ f(x) = f_+(x) - f_-(x)$. The integral of f is only defined if at least one of $\int f_+ < \infty$ and $\int f_- < \infty$ holds, in which case we define

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f_+ - \int_{\mathbb{R}^n} f_-.$$

Definition. A measurable function is called *integrable* if both $\int f_+ < \infty$ and $\int f_- < \infty$. Since $|f| = f_+ + f_-$, this is equivalent to $\int |f| < \infty$.

Properties of the Lebesgue Integral

(We will write $f \in L^1$ to mean f is measurable and $\int |f| < \infty$.)

(1) If $f, g \in L^1$ and $a, b \in \mathbb{R}$, then $af + bg \in L^1$, and $\int(af + bg) = a\int f + b\int g$. (We will write $f = g$ a.e. (almost everywhere) to mean $\lambda\{x : f(x) \neq g(x)\} = 0$.)

(2) If $f, g \in L^1$ and $f = g$ a.e., then $\int f = \int g$.

(3) If $f \geq 0$ and $\int f < \infty$, then $f < \infty$ a.e. Thus if $f \in L^1$, then $|f| < \infty$ a.e. (We will identify two functions if they agree a.e., e.g., $\chi_{\mathbb{Q}} = 0$ a.e.)

(4) If $f \geq 0$ and $\int f = 0$, then $f = 0$ a.e. (Not true if f can be both positive and negative, e.g., $\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx = 0$.)

(5) If A is measurable, $\int \chi_A = \lambda(A)$.

Definition. If A is a measurable set and $f : A \to [-\infty, \infty]$ is measurable, then $\int_A f = \int_{\mathbb{R}^n} f\chi_A$.

(6) If A and B are disjoint and $f\chi_{A\cup B} \in L^1$, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Definition. If $f : \mathbb{R}^n \to \mathbb{C}$ is measurable, and both $\mathcal{R}ef$ and $\mathcal{I}mf \in L^1$, define $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} \mathcal{R}ef + i\int_{\mathbb{R}^n} \mathcal{I}mf$.

(7) If $f : \mathbb{R}^n \to \mathbb{C}$ is measurable, then $\mathcal{R}ef$ and $\mathcal{I}mf \in L^1$ iff $|f| \in L^1$. Moreover, $|\int f| \leq \int |f|$.

Comparison of Riemann and Lebesgue integrals

If f is bounded and defined on a bounded set and f is Riemann integrable, then f is Lebesgue integrable and the two integrals are equal.

Theorem. If f is bounded and defined on a bounded set, then f is Riemann integrable iff f is continuous a.e.

Note: The two theories vary in their treatment of infinities (in both domain and range). For example, the improper Riemann integral $\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}dx$ exists and is finite, but $\frac{\sin x}{x}$ is *not* Lebesgue integrable over $[0,\infty)$ as $\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \infty$.

Convergence Theorems

If $\lim_{k\to\infty} f_k(x) = f(x)$ (maybe only a.e.), where f_k and f are measurable, then how are $\int f_k$ and $\int f$ related?

Examples.

(1) Let $f_k = \chi_{[k,\infty)}$. Then $f_k \geq 0$, $\lim f_k = 0$, and $\int f_k = \infty$, so $\lim \int f_k \neq \int \lim f_k$.

(2) Let $f_k = \chi_{[k,k+1]}$. Then again $\lim f_k = 0$, and $\int f_k = 1$, so $\lim \int f_k \neq \int \lim f_k$.

Monotone Convergence Theorem. (Jones calls "Increasing Conv. Thm.") If $0 \leq f_1 \leq f_2 \leq \cdots$ a.e., $f = \lim f_k$ a.e., and f_k and f are measurable, then $\lim_{k\to\infty}\int f_k = \int f$ $(= \int \lim_k f_k)$ (in $[0,\infty]$). Note: $\lim f_k$ exists a.e. by monotonicity.

Fatou's Lemma. If f_k are nonnegative a.e. and measurable, then

$$\int \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int f_k.$$

Lebesgue Dominated Convergence Theorem. Suppose $\{f_k\}$ is a sequence of complex-valued (or extended-real-valued) measurable functions. Assume $\lim_k f_k = f$ a.e., and assume that there exists a "dominating function," i.e., an *integrable* function g such that $|f_k(x)| \leq g(x)$ a.e. Then

$$\int f = \lim_{k\to\infty} \int f_k.$$

A corollary is the

Bounded Convergence Theorem. Let A be a measurable set of finite measure, and suppose $|f_k| \leq M$ on A. Assume $\lim_k f_k$ exists a.e. Then $\lim_k \int_A f_k = \int_A f$. (Apply Dominated Convergence Theorem with $g = M\chi_A$.)

Example. (using Fatou with a dominating sequence)

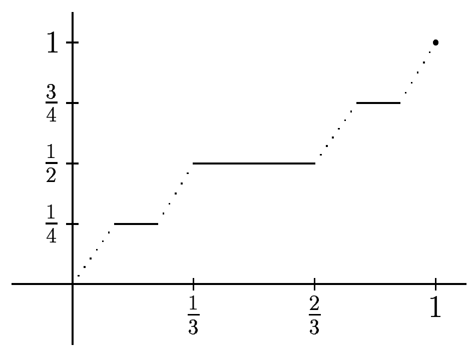
Extension of Lebesgue Dominated Convergence Theorem. Suppose $g_k \geq 0$, $g \geq 0$ are all integrable, and $\int g_k \to \int g$, and $g_k \to g$ a.e. Suppose f_k, f are all measurable, $|f_k| \leq g_k$

a.e. (which implies that f_k is integrable), and $f_k \to f$ a.e. (which implies $|f| \le g$ a.e.). Then $\int |f_k - f| \to 0$ (i.e. $\|f_k - f\|_{L^1} \to 0$).

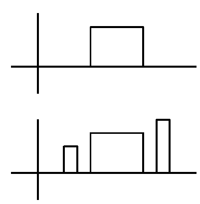
Proof. $|f_k - f| \le |f_k| + |f| \le g_k + g$ a.e. Apply Fatou to $g_k + g - |f_k - f|$ (which is ≥ 0 a.e.). Then $\int \liminf(g_k + g - |f_k - f|) \le \liminf \int (g_k + g - |f_k - f|)$. So $\int 2g \le \lim \int g_k + \int g - \limsup \int |f_k - f| = 2\int g - \limsup \int |f_k - f|$. Since $\int g < \infty$, $\limsup \int |f_k - f| \le 0$. Thus $\int |f_k - f| \to 0$. $\square$

Example — the Cantor Ternary Function

$f : [0,1] \to [0,1]$ is increasing and continuous. If $x \in C$ (the Cantor set), say $x = \sum_{k=1}^{\infty} \frac{d_k}{3^k}$ with $d_k \in \{0,2\}$, set $f(x) = \sum_{k=1}^{\infty} \frac{d_k/2}{2^k}$. On $[0,1]\backslash C$, map $\left(\frac{1}{2}, \frac{2}{3}\right)$ to $\frac{1}{2}$, $\left(\frac{1}{9}, \frac{2}{9}\right)$ to $\frac{1}{4}$, $\left(\frac{7}{9}, \frac{8}{9}\right)$ to $\frac{3}{4}$, etc. (For any $x \in [0,1]$, if $x = \sum_{k=1}^{\infty} \frac{d_k}{3^k}$ where $d_k \in \{0,1,2\}$, let K be the smallest k for which $d_k = 1$, and set $f(x) = \left(\sum_{k=1}^{K-1} \frac{d_k/2}{2^k}\right) + \frac{1}{2^K}$.)



Find $\int_0^1 f(x)dx$. Let
$$\begin{aligned}\varphi_1 &= \tfrac{1}{2}\chi_{\left(\frac{1}{3},\frac{2}{3}\right)}\\ \varphi_2 &= \varphi_1 + \tfrac{1}{4}\chi_{\left(\frac{1}{9},\frac{2}{9}\right)} + \tfrac{3}{4}\chi_{\left(\frac{7}{9},\frac{8}{9}\right)}\end{aligned}$$
etc.



Then each φ_k is simple (a finite linear comb of char. func. of meas. sets). (Note: if $\varphi = \sum_{j=1}^{N} a_j \chi_{A_j}$, where $A_j \in \mathcal{L}$ and $\lambda(A_j) < \infty$, then $\int \varphi = \sum_{j=1}^{N} a_j \lambda(A_j)$.) Also $\varphi_k(x) \to f(x)$ for $x \in [0,1]\backslash C$, and $\varphi_k(x) = 0$ for $x \in C (\forall\, k)$. Since $\lambda C = 0$, $\varphi_k \to f$ a.e. on $[0,1]$. So by MCT or LDCT or BCT, $\int_0^1 (f(x)dx = \lim_{k\to\infty} \int_0^1 \varphi_k(x)dx$. Now $\int \varphi_k = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3^2}\cdot\frac{1}{2^2}(1+3) + \frac{1}{3^3}\cdot\frac{1}{2^3}(1+3+5+7) + \cdots + \frac{1}{3^k}\cdot\frac{1}{2^k}(\underbrace{1+3+5+\cdots+(2^k-1)}_{(2^{k-1})^2 = 2^{2k-2}})$. (note: $1+3+5+\cdots+(2j-1) = j^2$) So $\int f = \lim_k \int \varphi_k = \sum_{m=1}^{\infty} \frac{1}{3^m}\frac{1}{2^m}2^{2m-2} = \frac{1}{6}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = \frac{1}{6}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{2}$.

"Multiple Integration" via Iterated Integrals

Suppose $n = m + l$, so $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^l$. For $x \in \mathbb{R}^n$, write $x = (y,z)$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^l$. Then $\int_{\mathbb{R}^n} f d\lambda_n = \int_{\mathbb{R}^n} f(x)d\lambda_n(x) = \int_{\mathbb{R}^n} f(y,z)d\lambda_n(y,z)$. Write dx for $d\lambda_n(x)$, dy for $d\lambda_m(y)$, dz for $d\lambda_l(z)$ (λ_n is Leb. meas. on $\mathbb{R}^n$). Consider the iterated integrals

$$\int_{\mathbb{R}^l}\left[\int_{\mathbb{R}^m} f(y,z)dy\right]dz \qquad \text{and} \qquad \int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^l} f(y,z)dz\right]dy.$$

Questions:

(1) When are these iterated integrals the same? (allowing change of order)

(2) When are they equal to $\int_{\mathbb{R}^n} f(x)dx$?

There are *two* key theorems, usually used in tandem.

① **Tonelli's Theorem**. Suppose $f \geq 0$ is measurable on $\mathbb{R}^n$. Then for a.e. $z \in \mathbb{R}^l$, the function $f_z(y) \equiv f(y,z)$ is measurable on $\mathbb{R}^m$ (as a function of y), and

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^l} \left[\int_{\mathbb{R}^m} f(y,z)dy\right] dz.$$

Example. Let $A \in \mathbb{R}^m$ be non-measurable. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(y,z) = \begin{cases} 0 & (z \neq z_0) \\ \chi_A(y) & (z = z_0) \end{cases}$. Then f is measurable on $\mathbb{R}^n$ (as $\lambda_n(\{x : f_n(x) \neq 0\}) = 0$). But the "slice function" $f_{z_0}(y) = f(y,z_0) = \chi_A(y)$ is not measurable on $\mathbb{R}^m$. However, since the set of z's for which $\int f_z(y)dy$ is undefined has measure zero, the iterated integral still makes sense and is 0.

② **Fubini's Theorem**. Suppose f is integrable on $\mathbb{R}^n$ (i.e., f meas., $\int |f| < \infty$). Then for a.e. $z \in \mathbb{R}^l$;, the slice functions $f_z(y)$ are integrable on $\mathbb{R}^m$, and

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^l} \left[\int_{\mathbb{R}^m} f(y,z)dy\right] dz.$$

Typical Case. Want to show $\int \left[\int f(y,z)dy\right] dz = \int \left[\int f(y,z)dz\right] dy$. Plan: Use Tonelli to verify hypothesis of Fubini:

(i) look at either iterated integral for $|f|$; show (if true) that the iterated integral is $< \infty$. Then by Tonelli, $\int_{\mathbb{R}^n} |f(x)|dx = \int \left[\int |f(y,z)|dz\right] dy < \infty$;

(ii) having verified now that $\int_{\mathbb{R}^n} |f| < \infty$, now apply Fubini.

Example. (Can't drop $\int |f| < \infty$ in Fubini.) Define f on $(0,1) \times (0,1)$ by $f(x,y) = \begin{cases} x^{-2} & 0 < y \leq x < 1 \\ -y^{-2} & 0 < x < y < 1 \end{cases}$. Then $\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \left(\frac{1}{x^2} - \int_x^1 \frac{dy}{y^2}\right) dx = \int_0^1 \left(\frac{1}{x} + 1 - \frac{1}{x}\right) dx = 1$. Similarly, $\int_0^1 \int_0^1 f(x,y)dxdy = -1$. Note that by Tonelli, $\int_{(0,1)\times(0,1)} |f(x,y)|d\lambda_2(x,y) = \int_0^1 \int_0^1 |f(x,y)|dydx = \int_0^1 \left(\frac{1}{x} + \int_x^1 \frac{dy}{y^2}\right) dx = \infty$.

L^p spaces

$1 \leq p < \infty$. Fix a measurable subset $A \subset \mathbb{R}^n$. Consider measurable functions $f : A \to \mathbb{C}$ for which $\int_A |f|^p < \infty$. Define $\|f\|_p = \left(\int_A |f|^p\right)^{\frac{1}{p}}$. On this set of functions, $\|f\|_p$ is only a *seminorm*:

$$\begin{aligned} \|f\|_p &\geq 0 (\text{but } \|f\|_p = 0 \text{ does not imply } f = 0, \text{only } f(x) = 0 \text{ a.e.}) \\ \|\alpha f\|_p &= |\alpha| \cdot \|f\|_p \\ \|f+g\|_p &\leq \|f\|_p + \|g\|_p (\text{Minkowski's Inequality}) \end{aligned}$$

(Note: $\|f\|_p = 0 \Rightarrow \int_A |f|^p = 0 \Rightarrow f = 0$ a.e. on A.) Define an equivalence relation on this set of functions:

$$f \sim g \text{ means } f = g \text{ a.e. on } A$$

Let $\widetilde{f} = \{g \text{ measurable on } A : f = g \text{ a.e.}\}$, the equivalence class of f. Define $\|\widetilde{f}\|_p = \|f\|_p$ (independent of choice of representative in $\widetilde{f}$). Define $L^p(A) = \{\widetilde{f} : \int_A |f|^p < \infty\}$. Then $\|\cdot\|_p$ is a *norm* on $L^p(A)$. We usually abuse notation and write $f \in L^p(A)$ (meaning $\widetilde{f} \in L^p(A)$).

Example. We say $f \in L^p(\mathbb{R}^n)$ is "continuous" if $\exists\, g \in L^p(\mathbb{R}^n)$ for which $g : \mathbb{R}^n \to \mathbb{C}$ is continuous and $f = g$ a.e. (and WLOG we can assume f is chosen to be that representative g of $\widetilde{f}$).

$p = \infty$. Fix $A^{\text{measurable}} \subset \mathbb{R}^n$. Consider "essentially bounded" measurable functions $f : A \to \mathbb{C}$, i.e., for which $\exists\, M < \infty$ so that $|f(x)| \le M$ a.e. on A. Define $\|f\|_\infty = \inf\{M : |f(x)| \le M \text{ a.e. on } A\}$, the *essential sup* of $|f|$. If $0 < \|f\|_\infty < \infty$, then for each $\varepsilon > 0$, $\lambda\{x \in A : |f(x)| > \|f\|_\infty - \varepsilon\} > 0$. As above, $\|\cdot\|_\infty$ is a seminorm on the set of essentially bounded meas. func's, and $\|\cdot\|_\infty$ is a norm on $L^\infty(A) = \{\widetilde{f} : \|f\|_\infty < \infty\}$.

Fact. For $f \in L^\infty(A)$, $|f(x)| \le \|f\|_\infty$ a.e. (Proof: $\{x : |f(x)| > \|f\|_\infty\} = \bigcup_{m=1}^{\infty} \{x : |f(x)| > \|f\|_\infty + \frac{1}{m}\}$. Each of these latter sets has measure 0.) So the inf is a min in the definition of $\|f\|_\infty$.

Fact. $L^\infty(\mathbb{R}^n)$ is *not* separable (i.e., it does not have a countable dense subset).

Example. For each $\alpha \in \mathbb{R}$, let $f_\alpha(x) = \chi_{[\alpha,\alpha+1]}(x)$. For $\alpha \ne \beta$, $\|f_\alpha - f_\beta\|_\infty = 1$. So $\{B_{\frac{1}{3}}(f_\alpha) : \alpha \in \mathbb{R}\}$ is an uncountable collection of disjoint nonempty open subsets in $L^\infty(\mathbb{R})$.

Conjugate Exponents. If $1 \le p \le \infty$, $1 \le q \le \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$ (where $\frac{1}{\infty} \equiv 0$), we say that p and q are *conjugate exponents*. Examples: $\begin{array}{c|cccc} p & 1 & 2 & 3 & \infty \\ q & \infty & 2 & \frac{3}{2} & 1 \end{array}$.

Hölder's Inequality. If $1 \le p \le \infty$, $1 \le q \le \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then $\int |fg| \le \|f\|_p \cdot \|g\|_q$. (Note: if $\int |fg| < \infty$, also $|\int fg| \le \int |fg| \le \|f\|_p \cdot \|g\|_q$.)

Remark. The cases $\left\{ \begin{array}{l} p = 1 \\ q = \infty \end{array} \right.$ and $\left\{ \begin{array}{l} p = \infty \\ q = 1 \end{array} \right.$ are obvious. When $p = 2$, $q = 2$, this is the Cauchy-Schwarz inequality $\int |fg| \le \|f\|_2 \cdot \|g\|_2$.

Completeness

Theorem. (Riesz-Fischer) Let $A^{\text{meas.}} \subset \mathbb{R}^n$, $1 \le p \le \infty$. Then $L^p(A)$ is complete (using $\|\cdot\|_p$).

Locally L^p Functions

Definition. Let $G^{\text{open}} \subset \mathbb{R}^n$. Define

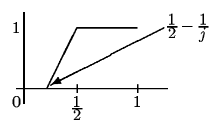
$$L^p_{\text{loc}}(G) = \{[\text{equiv classes of}] f : f^{\text{meas}} \text{ on } G, (\forall\, K^{\text{compact}} \subset G) f \in L^p(K)\}.$$

There is a metric on L^p_{loc} which makes it a complete metric space (but *not* a Banach space — not given by a norm). (**Fact**. If (X, ρ) is a metric space, and we define $\sigma(x, y) = \frac{\rho(x,y)}{1+\rho(x,y)}$, then σ is a metric on X, and (X, ρ) is uniformly equivalent to (X, σ). Use that $t \mapsto \frac{t}{1+t}$ is increasing on $[0, \infty)$ to show σ satisfies the triangle inequality. Note that $\sigma(x, y) < 1$ for all $x, y \in X$.)

Let $K_1, K_2, \ldots$ be a "compact exhaustion" of G, i.e., a sequence of nonempty compact subsets of G with $K_m \subset K_{m+1}^{o \leftarrow \text{interior}}$ and $\bigcup\limits_{m=1}^{\infty} K_m = G$ (e.g., $K_m = \{x \in G : \operatorname{dist}(x, G^C) \geq \frac{1}{m}$ and $|x| \leq m\}$). Then for any compact set $K \subset G$, $K \subset \bigcup\limits_{m=1}^{\infty} K_m \subset \bigcup\limits_{m=1}^{\infty} K_{m+1}^o$, so $\exists\, m$ for which $K \subset K_m$. The distance in $L^p_{\text{loc}}(G)$ is $d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f-g\|_{p,K_m}}{1+\|f-g\|_{p,K_m}}$. Clearly $f_j \to f$ in $L^p_{\text{loc}}(G)$ iff $(\forall\, K^{\text{compact}} \subset G) \|f_j - f\|_{p,K} \to 0$.

Continuous Functions not closed in L^p

Let $G \subset \mathbb{R}^n$ be open and *bounded*. Consider $C_b(G)$, the set of bounded continuous functions on G. Clearly $C_b(G) \subset L^p(G)$. But $C_b(G)$ is *not* closed in $L^p(G)$ $(p < \infty)$.

Example. $G = (0, 1)$ $\quad f_j =$ 

. Then $\{f_j\}$ is Cauchy in $\|\cdot\|_p$ for $1 \leq p < \infty$. But there is no cont. function f for which $\|f_j - f\|_p \to 0$ as $j \to \infty$.

Facts. Suppose $1 \leq p < \infty$ and $G^{\text{open}} \subset \mathbb{R}^n$.

(1) The set of simple functions (finite linear combinations of characteristic functions of measurable sets) with support in a bounded subset of G is dense in $L^p(G)$.

(2) The set of step functions (finite linear combinations of characteristic functions of rectangles) with support in a bounded subset of G is dense in $L^p(G)$.

(3) $C_C(G)$ is dense in $L^p(G)$

$\quad\hookrightarrow$ the set of continuous functions f whose support $\overline{\{x : f(x) \neq 0\}}$ is compact, $\subset G$.

(4) $C_C^{\infty}(G)$ is dense in $L^p(G)$ (idea: mollify a given $f \in L^p(G)$).

$\quad\hookrightarrow$ the set of C^{∞} functions whose support is a compact subset of G.

Consequence: For $1 \leq p < \infty$, $L^p(G)$ is separable (e.g., use (2), rectangles with rational endpoints, linear combinations with rational coefficients).

Example. Continuity of Translation in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ (uses $C_C(\mathbb{R}^n)$ dense). Let $f \in L^p(\mathbb{R}^n)$. For $y \in \mathbb{R}^n$, define $f_y(x) = f(x-y)$ (translate f by y).

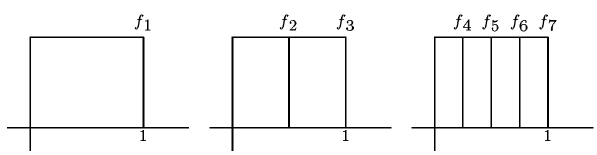
Claim. The map $y \mapsto f_y$ from $\mathbb{R}^n$ into $L^p(\mathbb{R}^n)$ is uniformly continuous.

Proof. Given $\varepsilon > 0$, choose $g \in C_C(\mathbb{R}^n)$ for which $\|g - f\|_p < \frac{\varepsilon}{3}$. Let $M = \lambda(\{x : g(x) \neq 0\}) < \infty$. By uniform continuity of g, $\exists\, \delta > 0$ for which $|z - y| < \delta \Rightarrow (\forall\, x)|g_z(x) - g_y(x)| < \frac{\varepsilon}{3(2M)^{\frac{1}{p}}}$. Then for $|z-y| < \delta$, $\|g_z - g_y\|_p^p = \int |g_z - g_y|^p \leq \lambda(\{x : g_z(x) \neq 0 \text{ or } g_y(x) \neq 0\}) \left(\frac{\varepsilon}{3(2M)^{\frac{1}{p}}}\right)^p \leq (2M)\frac{\varepsilon^p}{3^p(2M)}$, i.e., $\|g_z - g_y\|_p \leq \frac{\varepsilon}{3}$, and thus $\|f_z - f_y\|_p \leq \|f_z - g_z\|_p + \|g_z - g_y\|_p + \|g_y - f_y\|_p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. $\square$

L^p convergence and pointwise a.e. convergence

For $p = \infty$. $f_k \to f$ in $L^\infty \Rightarrow$ on the complement of a set of measure 0, $f_k \to f$ uniformly. (Let $A_k = \{x : |f_k(x) - f(x)| > \|f_k - f\|_\infty\}$, and $A = \bigcup_{k=1}^{\infty} A_k$. Since each $\lambda(A_k) = 0$, also $\lambda(A) = 0$. On A^C, $(\forall\, k)|f_k(x) - f(x)| \leq \|f_k - f\|_\infty$, so $f_k \to f$ unif. on A^C.)

For $1 \leq p < \infty$. Let $A^{\text{meas.}} \subset \mathbb{R}^n$. Here $f_k \to f$ in $L^p(A)$ (i.e., $\|f_k - f\|_p \to 0$) does not imply that $f_k \to f$ a.e. (Example. $A = [0,1]$, $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,\frac{1}{2}]}$, $f_3 = \chi_{[\frac{1}{2},1]}$, $f_4 = \chi_{[0,\frac{1}{4}]}, \cdots$



etc. Clearly $\|f_k\|_p \to 0$, so $f_k \to 0$ in L^p, but for no $x \in [0,1]$ does $f_k(x) \to 0$.) So L^p convergence for $1 \leq p < \infty$ does not imply a.e. convergence. However:

Fact. If $1 \leq p < \infty$ and $f_k \to f$ in $L^p(A)$, then $\exists$ a subsequence f_{k_j} for which $f_{k_j} \to f$ a.e. as $j \to \infty$.

Example. Suppose $A^{\text{meas}} \subset \mathbb{R}^n$, $1 \leq p < \infty$, $f_k, f \in L^p(A)$, and $f_k \to f$ a.e. Question: when does $f_k \to f$ in $L^p(A)$ (i.e. $\|f_k - f\|_p \to 0$)? Answer: In this situation, $f_k \to f$ in $L^p(A)$ iff $\|f_k\|_p \to \|f\|_p$.

Proof.

($\Rightarrow$) If $\|f_k - f\|_p \to 0$, then $|\|f_k\|_p - \|f\|_p| \leq \|f_k - f\|_p$, so $\|f_k\|_p \to \|f\|_p$.

($\Leftarrow$) First, observe: **Fact**. If $x, y \geq 0$, then $(x+y)^p \leq 2^p(x^p + y^p)$. (Proof: let $z = \max\{x, y\}$; then $(x+y)^p \leq (2z)^p = 2^p z^p \leq 2^p(x^p + y^p)$.) We will use Fatou's lemma with a "dominating sequence." We have

$$|f_k - f|^p \leq (|f_k| + |f|)^p \leq 2^p(|f_k|^p + |f|^p).$$

Apply Fatou to $2^p(|f_k|^p + |f|^p) - |f_k - f|^p \geq 0$. By assumption, $\|f_k\|_p \to \|f\|_p$, so $\int |f_k|^p \to \int |f|^p$. We thus get

$$\begin{aligned} \int 2^p(|f|^p + |f|^p - 0) &\leq \liminf \int 2^p(|f_k|^p + |f|^p) - |f_k - f|^p \\ &= \int 2^p|f|^p + \int 2^p|f|^p - \limsup |f_k - f|^p. \end{aligned}$$

Thus $\limsup \int |f_k - f|^p \leq 0$. So $\|f_k - f\|_p^p = \int |f_k - f|^p \to 0$. So $\|f_k - f\|_p \to 0$.

$\square$

Intuition for growth of functions in $L^p(\mathbb{R}^n)$

Fix n, fix p with $1 \leq p < \infty$, and fix a.

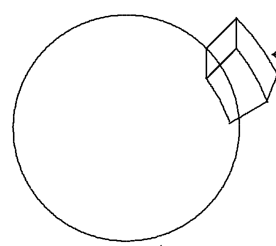
Let $f_1(x) = \frac{1}{|x|^a}\chi_{\{x:|x|<1\}}$ $\quad$ $f_2(x) = \frac{1}{|x|^a}\chi_{\{x:|x|>1\}}$

$\hookrightarrow$ blows up near $x = 0$ for $a > 0$, no problems as $|x| \to \infty$

$\hookrightarrow$ no problems near $x = 0$, investigate behavior as $|x| \to \infty$.

Polar Coordinates in $\mathbb{R}^n$

$rS^{n-1} = \{x : |x| = r\}$



$\leftarrow n -$ dim "volume element"

$dV = r^{n-1}dr\,d\sigma$

($d\sigma$ is "surface area" measure on S^{n-1})

So $\int_{\mathbb{R}^n} |f_1(x)|^p dx = \int_{S^{n-1}} \left[\int_0^1 \left(\frac{1}{r^a}\right)^p r^{n-1}dr\right] d\sigma = \omega_n \int_0^1 r^{n-ap-1}dr$ where $\omega_n = \sigma(S^{n-1})$. This is $< \infty$ iff $n - ap - 1 > -1$, i.e., $a < \frac{n}{p}$. So $f_1 \in L^p(\mathbb{R}^n)$ iff $a < \frac{n}{p}$. Similarly, $f_2 \in L^p(\mathbb{R}^n)$ iff $a > \frac{n}{p}$.

Conclusion. For any $p \neq q$ with $1 \leq p, q \leq \infty$, $L^p \not\subset L^q$. However, for sets A of finite measure, we have:

Claim. If $\lambda(A) < \infty$ and $1 \leq p < q \leq \infty$, then $L^q(A) \subset L^p(A)$, and

$$\|f\|_p \leq \lambda(A)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Proof. This is obvious when $q = \infty$. So suppose $1 \leq p < q < \infty$. Let $r = \frac{q}{p}$. Then $1 < r < \infty$. Let s be the conjugate exponent to r, so $\frac{1}{r} + \frac{1}{s} = 1$. Then $\frac{1}{s} = 1 - \frac{p}{q} = p\left(\frac{1}{p} - \frac{1}{q}\right)$. By Hölder,

$$\begin{aligned} \|f\|_p^p = \int_A |f|^p &= \int \chi_A |f|^p \leq \|\chi_A\|_s \cdot \| \, |f|^p \|_r = \lambda(A)^{\frac{1}{s}} \left(\int |f|^q \right)^{\frac{p}{q}} \\ &= \lambda(A)^{p\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_q^p. \end{aligned}$$

Take p^{th} roots. $\square$

Remark. This is in sharp contrast to what happens in l^p: For sequences $\{x_k\}_{k=1}^\infty$, the l^∞ norm is $\|x\|_\infty = \sup_k |x_k|$, and for $1 \leq p < \infty$, the l^p norm is $\|x\|_p = \left(\sum_k |x_k|^p\right)^{\frac{1}{p}}$.

Claim. For $1 \leq p < q \leq \infty$, $l^p \subset l^q$. In fact $\|x\|_q \leq \|x\|_p$.

Proof. Obvious when $q = \infty$. So suppose $1 \leq p < q < \infty$. Then

$$\begin{aligned} \|x\|_q^q = \sum_k |x_k|^q &= \sum_k |x_k|^{q-p} |x_k|^p \\ &\leq \|x\|_\infty^{q-p} \sum |x_k|^p \leq \|x\|_p^{q-p} \|x\|_p^p = \|x\|_p^q. \end{aligned}$$

Take q^{th} roots to get $\|x\|_q \leq \|x\|_p$. $\square$