Math 555 Linear Analysis
Winter 2009
Lecture Notes

Contents

Ordinary Differential Equations (ODEs)

ODEs

Let $\mathbb{F}$ be $\mathbb{R}$ on $\mathbb{C}$. Throughout this discussion, $|\cdot|$ will denote the Euclidean norm (i.e ℓ^2-norm) on $\mathbb{F}^n$ (so $\|\cdot\|$ is free to be used for norms on function spaces). An ODE is an equation of the form

$$g(t, x, x', \ldots, x^{(m)}) = 0$$

where g maps a subset of $\mathbb{R} \times (\mathbb{F}^n)^{m+1}$ into $\mathbb{F}^n$. A *solution* of this ODE on an interval $I \subset \mathbb{R}$ is a function $x : I \to \mathbb{F}^n$ for which $x', x'', \ldots, x^{(m)}$ exist at each $t \in I$, and

$$(\forall\, t \in I) \qquad g(t, x(t), x'(t), \ldots, x^{(m)}(t)) = 0\ .$$

We will focus on the case where $x^{(m)}$ can be solved for explicitly; i.e., the equation takes the form

$$x^{(m)} = f(t, x, x', \ldots, x^{(m-1)}),$$

and where the function f mapping a subset of $\mathbb{R} \times (\mathbb{F}^n)^m$ into $\mathbb{F}^n$ is continuous. This equation is called an m^{th}*-order* $n \times n$ system of ODE's. Note that if x is a solution defined on an interval $I \subset \mathbb{R}$ then the existence of $x^{(m)}$ on I (including one-sided limits at the endpoints of I) implies that $x \in C^{m-1}(I)$, and then the equation implies $x^{(m)} \in C(I)$, so $x \in C^m(I)$.

Reduction to First-Order Systems

Every m^{th}-order $n \times n$ system of ODE's is equivalent to a first-order $mn \times mn$ system of ODE's. Defining

$$y_j(t) = x^{(j-1)}(t) \quad \in \mathbb{F}^n \quad \text{for} \quad 1 \le j \le m$$

and

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \in \mathbb{F}^{mn},$$

the system

$$x^{(m)} = f(t, x, \ldots, x^{(m-1)})$$

is equivalent to the first-order $mn \times mn$ system

$$y' = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(t, y_1, \ldots, y_m) \end{bmatrix}.$$

Relabeling if necessary, we will focus on first-order $n \times n$ systems of the form $x' = f(t, x)$ where f maps a subset of $\mathbb{R} \times \mathbb{F}^n$ into $\mathbb{F}^n$ and f is continuous.

Example. Consider the $n \times n$ system $x'(t) = f(t)$ where $f : I \to \mathbb{F}^n$ is continuous on an interval $I \subset \mathbb{R}$. (Here f is independent of x.) Then calculus shows that for a fixed $t_0 \in I$, the general solution of the ODE (i.e., a form representing all possible solutions) is

$$x(t) = c + \int_{t_0}^{t} f(s)ds,$$

where $c \in \mathbb{F}^n$ is an arbitrary constant vector (i.e., $c_1, \ldots, c_n$ are n arbitrary constants in $\mathbb{F}$).

Provided f satisfies a Lipschitz condition (to be discussed soon), the general solution of a first-order system $x' = f(t, x)$ involves n arbitrary constants in $\mathbb{F}$ [or an arbitrary vector in $\mathbb{F}^n$] (whether or not we can express the general solution explicitly), so n scalar conditions [or one vector condition] must be given to specify a particular solution. For the example above, clearly giving $x(t_0) = x_0$ (for a known constant vector x_0) determines c, namely, $c = x_0$. In general, specifying $x(t_0) = x_0$ (these are called *initial conditions* (IC), even if t_0 is not the left endpoint of the t-interval I) determines a particular solution of the ODE.

Initial-Value Problems (IVP's) for First-order Systems

An IVP for the first-order system is the differential equation

$$DE: \qquad x' = f(t, x),$$

together with initial conditions

$$IC: \qquad x(t_0) = x_0\ .$$

A solution to the IVP is a solution $x(t)$ of the DE defined on an interval I containing t_0, which also satisfies the IC, i.e., for which $x(t_0) = x_0$.

Examples:

(1) Let $n = 1$. The solution of the IVP:

$$\begin{aligned} DE: &\qquad x' = x^2 \\ IC: &\qquad x(1) = 1 \end{aligned}$$

is $x(t) = \frac{1}{2-t}$, which blows up as $t \to 2$. So even if f is C^∞ on all of $\mathbb{R} \times \mathbb{F}^n$, solutions of an IVP do not necessarily exist for all time t.

(2) Let $n = 1$. Consider the IVP:

$$\begin{aligned} DE: &\qquad x' = 2\sqrt{|x|} \\ IC: &\qquad x(0) = 0 \ . \end{aligned}$$

For any $c \geq 0$, define $x_c(t) = 0$ for $t \leq c$ and $x_c(t) = (t-c)^2$ for $t \geq c$. Then every $x_c(t)$ for $c \geq 0$ is a solution of this IVP. So in general for continuous $f(t,x)$, IVP's may have non-unique solutions. (The difficulty here is that $f(t,x) = 2\sqrt{|x|}$ is not Lipschitz near $x = 0$.)

An Integral Equation Equivalent to an IVP

Suppose $x(t) \in C^1(I)$ is a solution of the IVP:

$$\begin{aligned} DE: &\qquad x' = f(t,x) \\ IC: &\qquad x(t_0) = x_0 \end{aligned}$$

defined on an interval $I \subset \mathbb{R}$ with $t_0 \in I$. Then for all $t \in I$,

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} x'(s)ds \\ &= x_0 + \int_{t_0}^{t} f(s,x(s))ds, \end{aligned}$$

so $x(t)$ is also a solution of the *integral equation*

(IE) $$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds \qquad (t \in I).$$

Conversely, suppose $x(t) \in C(I)$ is a solution of the integral equation (IE). Then $f(t,x(t)) \in C(I)$, so

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds \in C^1(I)$$

and $x'(t) = f(t,x(t))$ by the Fundamental Theorem of Calculus. So x is a C^1 solution of the DE on I, and clearly $x(t_0) = x_0$, so x is a solution of the IVP. We have shown:

Proposition. On an interval I containing t_0, x is a solution of the IVP: $DE: x' = f(t,x)$; $IC: x(t_0) = x_0$ (where f is continuous) with $x \in C^1(I)$ if and only if x is a solution of the integral equation (IE) on I with $x \in C(I)$.

The integral equation (IE) is a useful way to study the IVP. We can deal with the function space of continuous functions on I without having to be concerned about differentiability: continuous solutions of (IE) are automatically C^1. Moreover, the initial condition is built into the integral equation.

We will solve (IE) using a fixed-point formulation.

Definition. Let (X, d) be a metric space, and suppose $g : X \to X$. We say that g is a *contraction* [on X] if there exists $c < 1$ such that

$$(\forall\, x, y \in X) \qquad d(g(x), g(y)) \leq c d(x, y)$$

(c is sometimes called the contraction constant). A point $x_* \in X$ for which

$$g(x_*) = x_*$$

is called a *fixed point* of g.

Theorem.(**The Contraction Mapping Fixed-Point Theorem**)
Let (X, d) be a *complete* metric space and $g : X \to X$ be a contraction (with contraction constant $c < 1$). Then g has a unique fixed point $x_* \in X$. Moreover, for any $x_0 \in X$, if we generate the sequence $\{x_k\}$ iteratively by *functional iteration*

$$x_{k+1} = g(x_k) \quad \text{for} \quad k \geq 0$$

(sometimes called *fixed-point iteration*), then $x_k \to x_*$.

Proof. Fix $x_0 \in X$, and generate $\{x_k\}$ by $x_{k+1} = g(x_k)$ Then for $k \geq 1$,

$$d(x_{k+1}, x_k) = d(g(x_k), g(x_{k-1})) \leq c d(x_k, x_{k-1}).$$

By induction

$$d(x_{k+1}, x_k) \leq c^k d(x_1, x_0).$$

So for $n < m$,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq \left(\sum_{j=n}^{m-1} c^j \right) d(x_1, x_0) \\ &\leq \left(\sum_{j=n}^{\infty} c^j \right) d(x_1, x_0) = \frac{c^n}{1-c} d(x_1, x_0). \end{aligned}$$

Since $c^n \to 0$ as $n \to \infty$, $\{x_k\}$ is Cauchy. Since X is complete, $x_k \to x_*$ for some $x_k \in X$. Since g is a contraction, clearly g is continuous, so

$$g(x_*) = g(\lim x_k) = \lim g(x_k) = \lim x_{k+1} = x_*,$$

so x_* is a fixed point. If x and y are two fixed points of g in X, then

$$d(x, y) = d(g(x), g(y)) \leq c d(x, y),$$

so $(1 - c) d(x, y) \leq 0$, and thus $d(x, y) = 0$ and $x = y$. So g has a unique fixed point. □ □

Applications.

(1) Iterative methods for linear systems.

(2) *The Inverse Function Theorem* If $\Phi : N \to \mathbb{R}^n$ is a C^1 mapping on a neighborhood $N \subset \mathbb{R}^n$ of $x_0 \in \mathbb{R}^n$ satisfying $\Phi(x_0) = y_0$ and $\Phi'(x_0) \in \mathbb{R}^{n\times n}$ is invertible, then there exist neighborhoods $N_0 \subset N$ of x_0 and M_0 of y_0 and a C^1 mapping $\Psi : M_0 \to N_0$ for which $\Phi[N_0] = M_0$ and $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity mappings on M_0 and N_0, respectively.

Remark. Applying the Contraction Mapping Fixed-Point Theorem (C.M.F.-P.T.) to a function usually requires two steps:

(1) showing there is a topologically complete set S for which $g(S) \subset S$, and

(2) showing that g is a contraction on S.

To apply the C.M.F.-P.T. to the integral equation (IE), we need a further condition on the function $f(t,x)$.

Definition. Let $I \subset \mathbb{R}$ be an interval and $\Omega \subset \mathbb{F}^n$. We say that $f(t,x)$ mapping $I \times \Omega$ into $\mathbb{F}^n$ is *uniformly Lipschitz continuous with respect to* x if there is a constant L (called the *Lipschitz constant*) for which

$$(\forall\, t \in I)(\forall\, x, y \in \Omega) \qquad |f(t,x) - f(t,y)| \le L|x-y|\ .$$

We say that f is in (C, Lip) on $I \times \Omega$ if f is continuous on $I \times \Omega$ and f is uniformly Lipschitz continuous with respect to x on $I \times \Omega$.

For simplicity, we will consider intervals $I \subset \mathbb{R}$ for which t_0 is the left endpoint. Virtually identical arguments hold if t_0 is the right endpoint of I, or if t_0 is in the interior of I. (See Coddington & Levinson.)

Theorem (Local Existence and Uniqueness for (IE) for Lipschitz f)
Let $I = [t_0, t_0 + \beta]$ *and* $\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \le r\}$, *and suppose* $f(t,x)$ *is in* (C, Lip) *on* $I \times \Omega$. *Then there exists* $\alpha \in (0, \beta]$ *for which there is a unique solution of the integral equation*

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

in $C(I_\alpha)$ *where* $I_\alpha = [t_0, t_0 + \alpha]$. *Moreover, we can choose* α *to be any positive number satisfying*

$$\alpha \le \beta,\ \alpha \le \frac{r}{M}, \quad \text{and} \quad \alpha < \frac{1}{L}, \quad \text{where} \quad M = \max_{(t,x)\in I\times\Omega} |f(t,x)|$$

and L *is the Lipschitz constant for* f *in* $I \times \Omega$.

Proof. For any $\alpha \in (0, \beta]$, let $\|\cdot\|_\infty$ denote the max-norm on $C(I_\alpha)$:

$$\text{for} \quad x \in C(I_\alpha), \quad \|x\|_\infty = \max_{t_0 \le t \le t_0 + \alpha} |x(t)|\ .$$

Although this norm clearly depends on α, we do not include α in the notation. Let $\widetilde{x}_0$ denote the constant function $\widetilde{x}_0(t) \equiv x_0$ in $C(I_\alpha)$. For $\rho > 0$ let

$$X_{\alpha,\rho} = \{x \in C(I_\alpha) : \|x - \widetilde{x}_0\|_\infty \le \rho\}.$$

Then $X_{\alpha,\rho}$ is a complete metric space since it is a closed subset of the Banach space $(C(I_\alpha), \|\cdot\|_\infty)$. For any $\alpha \in (0, \beta]$, define $g : X_{\alpha,r} \to C(I_\alpha)$ by

$$(g(x))(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds :$$

g is well-defined on $X_{\alpha,r}$ and $g(x) \in C(I_\alpha)$ for $x \in X_{\alpha,r}$ since f is continuous on $I \times \overline{B_r(x_0)}$. Fixed points of g are solutions of the integral equation (IE).

Claim. Suppose $\alpha \in (0, \beta]$, $\alpha \le \frac{r}{M}$, and $\alpha < \frac{1}{L}$. Then g maps $X_{\alpha,r}$ into itself and g is a contraction on $X_{\alpha,r}$.

Proof of Claim: If $x \in X_{\alpha,r}$, then for $t \in I_\alpha$,

$$|(g(x))(t) - x_0| \le \int_{t_0}^{t} |f(s, x(s))|ds \le M\alpha \le r,$$

so $g : X_{\alpha,r} \to X_{\alpha,r}$. If $x, y \in X_{\alpha,r}$, then for $t \in I_\alpha$,

$$\begin{aligned} |(g(x))(t) - (g(y))(t)| &\le \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))|ds \\ &\le \int_{t_0}^{t} L|x(s) - y(s)|ds \\ &\le L\alpha\|x - y\|_\infty, \end{aligned}$$

so

$$\|g(x) - g(y)\|_\infty \le L\alpha\|x - y\|_\infty, \quad \text{and} \quad L\alpha < 1.$$

So by the C.M.F.-P.T., for α satisfying $0 < \alpha \le \beta$, $\alpha \le \frac{r}{M}$, and $\alpha < \frac{1}{L}$, g has a unique fixed point in $X_{\alpha,r}$, and thus the integral equation (IE) has a unique solution $x_*(t)$ in $X_{\alpha,r} = \{x \in C(I_\alpha) : \|x - \widetilde{x}_0\|_\infty \le r\}$. This is *almost* the conclusion of the Theorem, except we haven't shown x_* is the only solution in all of $C(I_\alpha)$. This uniqueness is better handled by techniques we will study soon, but we can still eke out a proof here. (We could say that f is only given on $I \times \overline{B_r(x_0)}$, but f can have a continuous extension to $I \times \mathbb{F}^n$.) Fix such an α. Then clearly for $0 < \gamma \le \alpha$, $x_*|_{I_\gamma}$ is the unique fixed point of g on $X_{\gamma,r}$. Suppose $y \in C(I_\alpha)$ is a solution of (IE) on I_α (using perhaps an extension of f) with $y \not\equiv x_\alpha$ on I_α. Let

$$\gamma_1 = \inf\{\gamma \in (0, \alpha] : y(t_0 + \gamma) \ne x_*(t_0 + \gamma)\}.$$

By continuity, $\gamma_1 < \alpha$. Since $y(t_0) = x_0$, continuity implies

$$\exists\, \gamma_0 \in (0, \alpha] \ni y|_{I_{\gamma_0}} \in X_{\gamma_0,r},$$

and thus $y(t) \equiv x_*(t)$ on I_{γ_0}. So $0 < \gamma_1 < \alpha$. Since $y(t) \equiv x_*(t)$ on I_{γ_1}, $y|_{I_{\gamma_1}} \in X_{\gamma_1,r}$. Let $\rho = M_{\gamma_1}$; then $\rho < M\alpha \leq r$. For $t \in I_{\gamma_1}$,

$$|y(t) - x_0| = |(g(y))(t) - x_0| \leq \int_{t_0}^{t} |f(s, y(s))| ds \leq M\gamma_1 = \rho,$$

so $y|_{I_{\gamma_1}} \in X_{\gamma_1,\rho}$. By continuity, there exists $\gamma_2 \in (\gamma_1, \alpha] \ni y|_{I_{\gamma_2}} \in X_{\gamma_1,r}$. But then $y(t) \equiv x_*(t)$ on I_{γ_2}, contradicting the definition of γ_1. □ □

The Picard Iteration

Although hidden in a few too many details, the main idea of the proof above is to study the convergence of functional iteration of g. If we choose the initial iterate to be $x_0(t) \equiv x_0$, we obtain the classical Picard iteration:

$$\begin{cases} x_0(t) & \equiv & x_0 \\ x_{k+1}(t) & = & x_0 + \int_{t_0}^{t} f(s, x_k(s)) ds \quad \text{for} \quad k \geq 0 \end{cases}$$

The argument in the proof of the C.M.F.-P.T. gives only *uniform* estimates of, e.g., $x_{k+1} - x_k$: $\|x_{k+1} - x_k\|_\infty \leq L\alpha \|x_k - x_{k+1}\|_\infty$, leading to the condition $\alpha < \frac{1}{L}$. For the Picard iteration (and other iterations of similar nature, e.g., for Volterra integral equations of the second kind), we can get better results using *pointwise* estimates of $x_{k+1} - x_k$. The condition $\alpha < \frac{1}{L}$ turns out to be unnecessary (we will see another way to eliminate this assumption when we study continuation of solutions). For the moment, we will set aside the uniqueness question and focus on existence.

Theorem (Picard Global Existence for (IE) for Lipschitz f) *Let $I = [t_0, t_0 + \beta]$, and suppose $f(t, x)$ is in (C, Lip) on $I \times \mathbb{F}^n$. Then there exists a solution $x_*(t)$ of the integral equation (IE) in $C(I)$.*

Theorem (Picard Local Existence for (IE) for Lipschitz f) *Let $I = [t_0, t_0 + \beta]$ and $\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq r\}$, and suppose $f(t, x)$ is in (C, Lip) on $I \times \Omega$. Then there exists a solution $x_*(t)$ of the integral equation (IE) in $C(I_\alpha)$ where $I_\alpha = [t_0, t_0 + \alpha]$, $\alpha = \min\left(\beta, \frac{r}{M}\right)$, and where $M = \max_{(t,x) \in I \times \Omega} |f(t, x)|$.*

Proofs. We prove the two theorems together. For the global theorem, let $X = C(I)$ (i.e., $C(I, \mathbb{F}^n)$), and for the local theorem, let

$$X = X_{\alpha,r} \equiv \{x \in C(I_\alpha) : \|x - x_0\|_\infty \leq r\}$$

as before (where $x_0(t) \equiv x_0$). Then the map

$$(g(x))(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$$

maps X into X in both cases, and X is complete. Let

$$x_0(t) \equiv x_0, \quad \text{and} \quad x_{k+1} = g(x_k) \quad \text{for} \quad k \geq 0.$$

Let

$$\begin{aligned} M_0 &= \max_{t\in I}|f(t,x_0)| \qquad \text{(global thm)}, \\ M_0 &= \max_{t\in I_\alpha}|f(t,x_0)| \qquad \text{(local thm)}. \end{aligned}$$

Then for $t \in I$ (global) or $t \in I_\alpha$ (local),

$$\begin{aligned} |x_1(t)-x_0| &\le \int_{t_0}^t |f(s,x_0)|ds \le M_0(t-t_0) \\ |x_2(t)-x_1(t)| &\le \int_{t_0}^t |f(s,x_1(s))-f(s,x_0(s))|ds \\ &\le L\int_{t_0}^t |x_1(s)-x_0(s)|ds \\ &\le M_0L\int_{t_0}^t (s-t_0)ds = \frac{M_0L(t-t_0)^2}{2!} \end{aligned}$$

By induction, suppose $|x_k(t)-x_{k-1}(t)| \le M_0L^{k-1}\frac{(t-t_0)^k}{k!}$. Then

$$\begin{aligned} |x_{k+1}(t)-x_k(t)| &\le \int_{t_0}^t |f(s,x_k(s))-f(s,x_{k-1}(s))|ds \\ &\le L\int_{t_0}^t |x_k(s)-x_{k-1}(s)|ds \\ &\le M_0L^k\int_{t_0}^t \frac{(s-t_0)^k}{k!}ds = M_0L^k\frac{(t-t_0)^{k+1}}{(k+1)!}. \end{aligned}$$

So

$$\begin{aligned} \sum_{k=0}^\infty |x_{k+1}(t)-x_k(t)| &\le \frac{M_0}{L}\sum_{k=0}^\infty \frac{(L(t-t_0))^{k+1}}{(k+1)!} \\ &= \frac{M_0}{L}(e^{L(t-t_0)}-1) \\ &\le \frac{M_0}{L}(e^{L\gamma}-1) \end{aligned}$$

where $\gamma = \beta$ (global) or $\gamma = \alpha$ (local). Hence the series $x_0 + \sum_{k=0}^\infty (x_{k+1}(t)-x_k(t))$, which has x_{N+1} as its N^{th} partial sum, converges absolutely and uniformly on I (global) or I_α (local) by the Weierstrass M-test. Let $x_*(t) \in C(I)$ (global) or $\in C(I_\alpha)$ (local) be the limit function. Since

$$|f(t,x_k(t))-f(t,x_*(t))| \le L|x_k(t)-x_*(t)|,$$

$f(t,x_k(t))$ converges uniformly to $f(t,x_*(t))$ on I (global) or I_α (local), and thus

$$\begin{aligned} g(x_*)(t) &= x_0 + \int_{t_0}^t f(s,x_*(s))ds \\ &= \lim_{k\to\infty}(x_0 + \int_{t_0}^t f(s,x_k(x))ds) \\ &= \lim_{k\to\infty} x_{k+1}(t) = x_*(t), \end{aligned}$$

for all $t \in I$ (global) or I_α (local). Hence $x_*(t)$ is a fixed point of g in X, and thus also a solution of the integral equation (IE) in $C(I)$ (global) or $C(I_\alpha)$ (local.) □

Corollary. The solution $x_*(t)$ of (IE) satisfies

$$|x_*(t) - x_0| \leq \frac{M_0}{L}(e^{L(t-t_0)} - 1)$$

for $t \in I$ (global) or $t \in I_\alpha$ (local), where $M_0 = \max_{t \in I} |f(t, x_0)|$ (global), $M_0 = \max_{t = I_\alpha} |f(t, x_0)|$ (local).

Proof. This is established in the proof above. □ □

Remark. In each of the statements of the last three Theorems, we could replace "solution of the integral equation (IE)" with "solution of the IVP: $DE : x' = f(t, x)$; $IC : x(t_0) = x_0$" because of the equivalence of these two problems.

Examples.

(1) Consider a *linear* system $x' = A(t)x + b(t)$, where $A(t) \in \mathbb{C}^{n \times n}$ and $b(t) \in \mathbb{C}^n$ are in $C(I)$ (where $I = [t_0, t_0 + \beta]$). Then f is in (C, Lip) on $I \times \mathbb{F}^n$:

$$|f(t, x) - f(t, y)| \leq |A(t)x - A(t)y| \leq \left(\max_{t \in I} \|A(t)\| \right) |x - y|.$$

Hence there is a solution of the IVP: $x' = A(t)x + b(t)$, $x(t_0) = x_0$ in $C^1(I)$.

(2) $(n = 1)$ Consider the IVP: $x' = x^2$, $x(0) = x_0 > 0$. Then $f(t, x) = x^2$ is not in (C, Lip) on $I \times \mathbb{R}$. It is, however, in (C, Lip) on $I \times \Omega$ where $\Omega = \overline{B_r(x_0)} = [x_0 - r, x_0 + r]$ for each fixed r. For a given $r > 0$, $M = (x_0 + r)^2$, and $\alpha = \frac{r}{M} = \frac{r}{(x_0+r)^2}$ in the local theorem is maximized for $r = x_0$, where $\alpha = \frac{1}{4x_0}$. So the local theorem guarantees a solution in $\left[0, \frac{1}{4x_0}\right]$. The actual solution $x_*(t) = (x_0^{-1} - t)^{-1}$ exists in $\left[0, \frac{1}{x_0}\right)$.

Local Existence for Continuous f

A condition similar to the Lipschitz condition is needed to guarantee that the Picard iterates converge; it is also needed for uniqueness, which we will return to shortly. It is, however, still possible to prove a local existence theorem assuming only that f is continuous, without assuming the Lipschitz condition. We will need the following form of Ascoli's Theorem:

Theorem (Ascoli) *Let X and Y be metric spaces with X compact. Let $\{f_k\}$ be an* equicontinuous *sequence of functions $f_k : X \to Y$, i.e.,*

$$(\forall \varepsilon > 0)(\exists \delta > 0) \quad \textit{such that} \quad (\forall k \geq 1)(\forall x_1, x_2 \in X)$$
$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f_k(x_1), f_k(x_2)) < \varepsilon$$

(in particular, each f_k is continuous), and suppose for each $x \in X$, $\overline{\{f_k(x) : k \geq 1\}}$ is a compact subset of Y. Then there is a subsequence $\{f_{k_j}\}_{j=1}^\infty$ and a continuous $f : X \to Y$ such that

$$f_{k_j} \to f \quad \textit{uniformly on } X.$$

Remark. If $Y = \mathbb{F}^n$, the condition $(\forall\, x \in X)$ $\overline{\{f_k(x) : k \geq 1\}}$ is compact is equivalent to the sequence $\{f_k\}$ being *pointwise bounded*, i.e.,

$$(\forall\, x \in X)(\exists\, M_x) \quad \text{such that} \quad (\forall\, k \geq 1) \quad |f_k(x)| \leq M_x.$$

Example. Suppose $f_k : [a,b] \to \mathbb{R}$ is a sequence of C^1 functions, and suppose there exists $M > 0$ such that

$$(\forall\, k \geq 1) \quad \|f_k\|_\infty + \|f_k'\|_\infty \leq M$$

(where $\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$). Then for $a \leq x_1 < x_2 \leq b$,

$$|f_k(x_2) - f_k(x_1)| \leq \int_{x_1}^{x_2} |f_k'(x)| dx \leq M|x_2 - x_1|,$$

so $\{f_k\}$ is equicontinuous (take $\delta = \frac{\varepsilon}{M}$), and $\|f_k\|_\infty \leq M$ certainly implies $\{f_k\}$ is pointwise bounded. So by Ascoli's Theorem, some subsequence of $\{f_k\}$ converges uniformly to a continuous function $f : [a,b] \to \mathbb{R}$.

Theorem. The Cauchy-Peano Existence Theorem
Let $I = [t_0, t_0 + \beta]$ and $\Omega = \overline{B_r(x_0)} = \{x \in \mathbb{F}^n : |x - x_0| \leq r\}$, and suppose $f(t,x)$ is continuous on $I \times \Omega$. Then there exists a solution $x_*(t)$ of the integral equation

$$\text{(IE)} \qquad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds$$

in $C(I_\alpha)$ where $I_\alpha = [t_0, t_0 + \alpha]$, $\alpha = \min\left(\beta, \frac{r}{M}\right)$, and $M = \max_{(t,x) \in I \times \Omega} |f(t,x)|$ (and thus $x_*(t)$ is a C^1 solution of the IVP: $x' = f(t,x)$; $x(t_0) = x_0$ in I_α).

Proof. The idea of the proof is to construct continuous approximate solutions explicitly (we will use the piecewise linear interpolants of grid functions generated by Euler's method), and use Ascoli's Theorem to take the uniform limit of some subsequence. For each integer $k \geq 1$, define $x_k(t) \in C(I_\alpha)$ as follows: partition $[t_0, t_0 + \alpha]$ into k equal subintervals (for $0 \leq \ell \leq k$, let $t_\ell = t_0 + \ell \frac{\alpha}{k}$ (note: t_ℓ depends on k too)), set $x_k(t_0) = x_0$, and for $\ell = 1, 2, \ldots, k$ define $x_k(t)$ in $(t_{\ell-1}, t_\ell]$ inductively by $x_k(t) = x_k(t_{\ell-1}) + f(t_{\ell-1}, x_k(t_{\ell-1}))(t - t_{\ell-1})$. For this to be well-defined we must check that $|x_k(t_{\ell-1}) - x_0| \leq r$ for $2 \leq \ell \leq k$ (it is obvious for $\ell = 1$); inductively, we have

$$\begin{aligned}
|x_k(t_{\ell-1}) - x_0| &\leq \sum_{i=1}^{\ell-1} |x_k(t_i) - x_k(t_{i-1})| \\
&= \sum_{i=1}^{\ell-1} |f(t_{i-1}, x_k(t_{i-1}))| \cdot |t_i - t_{i-1}| \\
&\leq M \sum_{i=1}^{\ell-1} (t_i - t_{i-1}) \\
&= M(t_{\ell-1} - t_0) \leq M\alpha \leq r
\end{aligned}$$

by the choice of α. So $x_k(t) \in C(I_\alpha)$ is well defined. A similar estimate shows that for $t, \tau \in [t_0, t_0 + \alpha]$,

$$|x_k(t) - x_k(\tau)| \le M|t - \tau|.$$

This implies that $\{x_k\}$ is equicontinuous; it also implies that

$$(\forall\, k \ge 1)(\forall\, t \in I_\alpha) \quad |x_k(t) - x_0| \le M\alpha \le r,$$

so $\{x_k\}$ is pointwise bounded (in fact, uniformly bounded). So by Ascoli's Theorem, there exists $x_*(t) \in C(I_\alpha)$ and a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ converging uniformly to $x_*(t)$. It remains to show that $x_*(t)$ is a solution of (IE) on I_α. Since each $x_k(t)$ is continuous and piecewise linear on I_α,

$$x_k(t) = x_0 + \int_{t_0}^{t} x_k'(s)ds$$

(where $x_k'(t)$ is piecewise constant on I_α and is defined for all t except t_ℓ $(1 \le \ell \le k-1)$, where we define it to be $x_k'(t_\ell^+)$). Define

$$\Delta_k(t) = x_k'(t) - f(t, x_k(t)) \quad \text{on} \quad I_\alpha$$

(note that $\Delta_k(t_\ell) = 0$ for $0 \le \ell \le k-1$ by definition). We claim that $\Delta_k(t) \to 0$ uniformly on I_α as $k \to \infty$. Indeed, given k, we have for $1 \le \ell \le k$ and $t \in (t_{\ell-1}, t_\ell)$ (including t_k if $\ell = k$), that

$$|x_k'(t) - f(t, x_k(t))| = |f(t_{\ell-1}, x_k(t_{\ell-1})) - f(t, x_k(t))|.$$

Noting that $|t - t_{\ell-1}| \le \frac{\alpha}{k}$ and

$$|x_k(t) - x_k(t_{\ell-1})| \le M|t - t_{\ell-1}| \le M\frac{\alpha}{k},$$

the uniform continuity of f (being continuous on the compact set $I \times \Omega$) implies that

$$\max_{t \in I_\alpha} |\Delta_k(t)| \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, in particular, $\Delta_{k_j}(t) \to 0$ uniformly on I_α. Now

$$\begin{aligned} x_{k_j}(t) &= x_0 + \int_{t_0}^{t} x_{k_j}'(s)ds \\ &= x_0 + \int_{t_0}^{t} f(s, x_{k_j}(s))ds + \int_{t_0}^{t} \Delta_{k_j}(s)ds. \end{aligned}$$

Since $x_{k_j} \to x_*$ uniformly on I_α, the uniform continuity of f on $I \times \Omega$ now implies that $f(t, x_{k_j}(t)) \to f(t, x_*(t))$ uniformly on I_α, so taking the limit as $j \to \infty$ on both sides of this equation for each $t \in I_\alpha$, we obtain that x_* satisfies (IE) on I_α □ □

Remark. In general, the choice of a subsequence of $\{x_k\}$ is necessary: there are examples where the sequence $\{x_k\}$ does not converge. (See Problem 12, Chapter 1 of Coddington & Levinson.)

Uniqueness

Uniqueness theorems are typically proved by comparison theorems for solutions of scalar differential equations, or by inequalities. The most fundamental of these inequalities is Gronwall's inequality, which applies to real first-order linear scalar equations.

Recall that a first-order linear scalar initial value problem

$$IVP: \qquad u' = a(t)u + b(t), \quad u(t_0) = u_0$$

can be solved by multiplying by the integrating factor $e^{-\int_{t_0}^t a}$ (i.e., $e^{-\int_{t_0}^t a(s)ds}$), and then integrating from t_0 to t. That is,

$$\frac{d}{dt}\left(e^{-\int_{t_0}^t a}u(t)\right) = e^{-\int_{t_0}^t a}b(t),$$

implies that

$$\begin{aligned} e^{-\int_{t_0}^t a}u(t) - u_0 &= \int_{t_0}^t \frac{d}{ds}\left(e^{-\int_{t_0}^s a}u(s)\right)ds \\ &= \int_{t_0}^t e^{-\int_{t_0}^s a}b(s)ds \end{aligned}$$

which in turn implies that

$$u(t) = u_0 e^{\int_{t_0}^t a} + \int_{t_0}^t e^{\int_s^t a}b(s)ds.$$

Since $f(t) \le g(t)$ on $[c,d]$ implies $\int_c^d f(t)dt \le \int_c^d g(t)dt$, the identical argument with "=" replaced by "$\le$" gives

Theorem (Gronwall's Inequality - differential form) *Let $I = [t_0, t_1]$. Suppose $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ are continuous, and suppose $u : I \to \mathbb{R}$ is in $C^1(I)$ and satisfies*

$$u'(t) \le a(t)u(t) + b(t) \quad \textit{for} \quad t \in I, \quad \textit{and} \quad u(t_0) = u_0.$$

Then

$$u(t) \le u_0 e^{\int_{t_0}^t a} + \int_{t_0}^t e^{\int_s^t a}b(s)ds.$$

Remarks.

(1) Thus a solution of the differential inequality is bounded above by the solution of the equality (i.e., differential equation $u' = au + b$).

(2) The result clearly still holds if u is only continuous and piecewise C^1, and $a(t)$ and $b(t)$ are only piecewise continuous.

(3) There is also an integral form of Gronwall's inequality (i.e., the hypothesis is an integral inequality): if $\varphi, \psi, \alpha \in C(I)$ are real-valued with $\alpha \geq 0$ on I, and

$$\varphi(t) \leq \psi(t) + \int_{t_0}^{t} \alpha(s)\varphi(s)ds \quad \text{for} \quad t \in I,$$

then

$$\varphi(t) \leq \psi(t) + \int_{t_0}^{t} e^{\int_s^t \alpha}\alpha(s)\psi(s)ds.$$

In particular, if $\psi(t) \equiv c$ (a constant), then $\varphi(t) \leq ce^{\int_{t_0}^t \alpha}$. (The differential form is applied to the C^1 function $u(t) = \int_{t_0}^t \alpha(s)\varphi(s)ds$ in the proof — see problem 4 on Prob. Set. 9.)

(4) For $a(t) \geq 0$, the differential form is also a consequence of the integral form: integrating

$$u' \leq a(t)u + b(t) \quad \text{from} \quad t_0 \quad \text{to} \quad t$$

gives

$$u(t) \leq \psi(t) + \int_{t_0}^{t} a(s)u(s)ds,$$

where

$$\psi(t) = u_0 + \int_{t_0}^{t} b(s)ds,$$

so integration by parts gives

$$\begin{aligned} u(t) &\leq \psi(t) + \int_{t_0}^{t} e^{\int_s^t a}a(s)\psi(s)ds \\ &= \cdots = u_0 e^{\int_{t_0}^t a} + \int_{t_0}^{t} e^{\int_s^t a}b(s)ds. \end{aligned}$$

(5) Caution: a differential inequality implies an integral inequality, but *not* vice versa: $f \leq g \not\Rightarrow f' \leq g'$.

(6) The integral form doesn't require $\varphi \in C^1$ (just $\varphi \in C(I)$), but is restricted to $\alpha \geq 0$. The differential form has no sign restriction on $a(t)$, but it requires a stronger hypothesis (in view of (5) and the requirement that u be continuous and piecewise C^1).

Uniqueness for Locally Lipschitz f

We start with a one-sided local uniqueness theorem for the initial value problem

$$IVP: \qquad x' = f(t,x); \quad x(t_0) = x_0.$$

Theorem. *Suppose for some $\alpha > 0$ and some $r > 0$, $f(t,x)$ is in (C, Lip) on $I_\alpha \times \overline{B_r(x_0)}$, and suppose $x(t)$ and $y(t)$ both map I_α into $\overline{B_r(x_0)}$ and both are C^1 solutions of (IVP) on $I_\alpha = [t_0, t_0 + \alpha]$. Then $x(t) = y(t)$ for $t \in I_\alpha$.*

Proof. Set

$$u(t) = |x(t) - y(t)|^2 = \langle x(t) - y(t), x(t) - y(t) \rangle$$

(in the Euclidean inner product on $\mathbb{F}^n$). Then $u : I_\alpha \to [0, \infty)$ and $u \in C^1(I_\alpha)$ and for $t \in I_\alpha$,

$$\begin{aligned} u' &= \langle x - y, x' - y' \rangle + \langle x' - y', x - y \rangle \\ &= 2\mathcal{R}e\langle x - y, x' - y' \rangle \leq 2|\langle x - y, x' - y' \rangle| \\ &= 2|\langle x - y, (f(t,x) - f(t,y)) \rangle| \\ &\leq |x - y| \cdot |f(t,x) - f(t,y)| \\ &\leq 2L|x - y|^2 = 2Lu \ . \end{aligned}$$

Thus $u' \leq Lu$ on I_α and $u(t_0) = x(t_0) - y(t_0) = x_0 - x_0 = 0$. By Gronwall's inequality, $u(t) \leq u_0 e^{Lt} = 0$ on I_α, so since $u(t) \geq 0$, $u(t) \equiv 0$ on I_α. □ □

Corollary.

(i) The same result holds if $I_\alpha = [t_0 - \alpha, t_0]$.

(ii) The same result holds if $I_\alpha = [t_0 - \alpha, t_0 + \alpha]$.

Proof. For (i), let $\widetilde{x}(t) = x(2t_0 - t)$, $\widetilde{y}(t) = y(2t_0 - t)$, and $\widetilde{f}(t,x) = -f(2t_0 - t, x)$. Then $\widetilde{f}$ is in (C, Lip) on $[t_0, t_0 + \alpha] \times \overline{B_r(x_0)}$, and $\widetilde{x}$ and $\widetilde{y}$ both satisfy the IVP

$$x' = \widetilde{f}(t,x); \quad x'(t_0) = x_0 \quad \text{on} \quad [t_0, t_0 + \alpha].$$

So by the Theorem, $\widetilde{x}(t) = \widetilde{y}(t)$ for $t \in [t_0, t_0 + \alpha]$, i.e., $x(t) = y(t)$ for $t \in [t_0 - \alpha, t_0]$. Now (ii) follows immediately by applying the Theorem in $[t_0, t_0 + \alpha]$ and applying (ii) in $[t_0 - \alpha, t_0]$.
□ □

Remark. The idea used in the proof of (i) is often called "time-reversal." The important part is that $\widetilde{x}(t) = x(c - t)$, for some constant c, so that $\widetilde{x}'(t) = -x'(c - t)$, etc. The choice of $c = 2t_0$ is convenient but not essential.

The main uniqueness theorem is easiest to state in its two-sided version (i.e., where t_0 is in the interior of the interval of definition of a solution of the IVP). One-sided versions (where t_0 is the left endpoint or right endpoint of the interval of definition of a solution of the IVP) are true and have the same proof, but require a more delicate statement. (Exercise: State one-sided theorems corresponding to the upcoming theorem precisely.)

Definition. Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$. We say that $f(t,x)$ mapping $\mathcal{D}$ into $\mathbb{F}^n$ is *locally Lipschitz continuous with respect to* x if for each $(t_1, x_1) \in \mathcal{D}$ there exists

$$\alpha > 0, \quad r > 0, \quad \text{and} \quad L > 0$$

for which $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D}$ and

$$(\forall\, t \in [t_1 - \alpha, t_1 + \alpha])(\forall\, x, y \in \overline{B_r(x_1)}) \quad |f(t,x) - f(t,y)| \leq L|x - y|$$

(i.e., f is uniformly Lipschitz continuous with respect to x in $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)}$). We will say $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ (not a standard notation) on $\mathcal{D}$ if f is continuous on $\mathcal{D}$ and locally Lipschitz continuous with respect to x on $\mathcal{D}$.

Example. Let $\mathcal{D}$ be an open set of $\mathbb{R} \times \mathbb{F}^n$. Suppose $f(t,x)$ maps $\mathcal{D}$ into $\mathbb{F}^n$, f is continuous on $\mathcal{D}$, and for $1 \leq i, j \leq n$, $\frac{\partial f_i}{\partial x_j}$ exists and is continuous in $\mathcal{D}$. (Briefly, we say f is continuous on $\mathcal{D}$ and C^1 with respect to x on $\mathcal{D}$.) Then $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$. (Exercise.)

Main Uniqueness Theorem. *Let $\mathcal{D}$ be an open set in $\mathbb{R} \times \mathbb{F}^n$, and suppose $f \in (C, \mathrm{Lip}_{\mathrm{loc}})$ on $\mathcal{D}$. Suppose $(t_0, x_0) \in \mathcal{D}$, $I \subset \mathbb{R}$ is some interval containing t_0 (which may be open or closed at either end), and suppose $x(t)$ and $y(t)$ are both solutions of the initial value problem*

$$IVP: \qquad x' = f(t,x): \quad x(t_0) = y_0$$

in $C^1(I)$ which satisfy $(t, x(t)) \in \mathcal{D}$ and $(t, y(t)) \in \mathcal{D}$ for $t \in I$. Then $x(t) \equiv y(t)$ on I.

Proof. We first show $x(t) \equiv y(t)$ on $\{t \in I : t \geq t_0\}$. If not, let

$$t_1 = \inf\{t \in I : t \geq t_0 \quad \text{and} \quad x(t) \neq y(t)\}.$$

Then $x(t) = y(t)$ on $[t_0, t_1)$ so by continuity $x(t_1) = y(t_1)$ (if $t_1 = t_0$, this is obvious). By continuity and the openness of $\mathcal{D}$ (as $(t_1, x(t_1)) \in \mathcal{D}$), there exist $\alpha > 0$ and $r > 0$ such that $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)} \subset \mathcal{D}$, f is uniformly Lipschitz continuous with respect to x in $[t_1 - \alpha, t_1 + \alpha] \times \overline{B_r(x_1)}$, and $x(t) \in \overline{B_r(x_1)}$ and $y(t) \in \overline{B_r(x_1)}$ for all t in $I \cap [t_1 - \alpha, t_1 + \alpha]$. By the previous theorem, $x(t) \equiv y(t)$ in $I \cap [t_1 - \alpha, t_1 + \alpha]$, contradicting the definition of t_1. Hence $x(t) \equiv y(t)$ on $\{t \in I : t \geq t_0\}$. Similarly, $x(t) \equiv y(t)$ on $\{t \in I : t \leq t_0\}$. Hence $x(t) \equiv y(t)$ on I. $\square$ $\square$

Remark. t_0 is allowed to be the left or right endpoint of I.