Assignment 8. Due **Fri., Dec. 5.**

Reading: Class Notes, pp. 61–95
Kato, Ch. 1 secs. 4-5
Horn and Johnson, ch. 2.6; 5.8; 7.0–7.4, 7.7.

1. (Iterative Methods for Solving $Ax = b$.) Suppose $A \in \mathbf{C}^{n\times n}$ is invertible and $b \in \mathbf{C}^n$ is given, and we want to solve the linear system $Ax = b$ for $x \in \mathbf{C}^n$. A *splitting method* writes A as $A = M - N$, where M is invertible (and linear systems $Mz = r$ are easy to solve), and given $x_0 \in \mathbf{C}^n$ generates a sequence of approximations $\{x_k\}$ satisfying $Mx_{k+1} = Nx_k + b$, $k = 0, 1, \ldots$.

   (a) Show that the sequence $x_k \to A^{-1}b$ for every choice of x_0 if and only if $\rho(M^{-1}N) < 1$. (Hint: Since the solution $x_* \equiv A^{-1}b$ satisfies $Mx_* = Nx_* + b$, subtract this equation from that defining x_{k+1} to see that the error $e_{k+1} \equiv x_* - x_{k+1}$ satisfies $e_{k+1} = (M^{-1}N)e_k$.)

   (b) If A has nonzero diagonal entries, then one possible choice for M is $M = \text{diag}(A)$. This is called the *Jacobi iteration*. Show that if A is *strictly (row) diagonally dominant* (i.e., for $1 \leq i \leq n$, $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$), then A is invertible, and for any given x_0, the Jacobi iteration generates a sequence $\{x_k\}$ which converges to $A^{-1}b$. (Hint: Show that $|||M^{-1}N|||_\infty < 1$ in the operator norm $|||\cdot|||_\infty$ on $\mathbf{C}^{n\times n}$ induced by the ℓ^∞ norm on $\mathbf{C}^n$.)

2. If $A, B \in \mathbf{C}^{n\times n}$ are Hermitian, we say that $A \geq B$ if $A - B$ is positive semidefinite.

   (a) Let $A \in \mathbf{C}^{n\times n}$ be Hermitian and $\alpha \geq 0$. Show that the spectral norm of A satisfies $\|A\| \leq \alpha$ if and only if $-\alpha I \leq A \leq \alpha I$.

   (b) If A is Hermitian and $\alpha I \leq A \leq \beta I$, and if $p(t)$ is a polynomial for which $p(t) \geq 0$ on $[\alpha, \beta]$, show that $p(A) \geq 0$.

   (c) If A is Hermitian and $\alpha I \leq A \leq \beta I$, and if $p(t)$ is a polynomial for which $p(t) \neq 0$ on $[\alpha, \beta]$, show that $p(A)$ is invertible.

3. Find the SVD for the matrix

$$A = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & 4 \end{pmatrix}.$$

4. **(a)** Prove the following minimax characterization of the singular values $\sigma_1 \geq \ldots \geq \sigma_n$ of $A \in \mathbf{C}^{m\times n}$ (with $m \geq n$): for $1 \leq k \leq n$,

$$\sigma_k = \min_{S_{n-k+1}} \left(\max_{\substack{x\in S_{n-k+1} \\ x\neq 0}} \frac{\|Ax\|}{\|x\|} \right),$$

where the min is taken over all subspaces S_{n-k+1} of dimension $n-k+1$, and $\|\cdot\|$ denotes the Euclidean norm.

(b) Use (a) to prove that if $A, B \in \mathbf{C}^{m\times n}$ have singular values $\sigma_1 \geq \ldots \geq \sigma_n$ and $\tau_1 \geq \ldots \geq \tau_n$, then for $1 \leq k \leq n$, $|\sigma_k - \tau_k| \leq \|A - B\|$, where $\|\cdot\|$ is the Euclidean operator norm.

5. Problem 5(b) shows that if a matrix is perturbed slightly, then its singular values can change at most by the (Euclidean operator) norm of the perturbation.

   (a) Show that this result fails drastically for eigenvalues by considering the perturbation

$$A_\epsilon = \begin{pmatrix} 0 & 1 & & & \\ \vdots & \ddots & \ddots & & \\ 0 & & \ddots & 1 \\ \epsilon & & & 0 \end{pmatrix}$$

   of $A_0 \in \mathbf{C}^{n\times n}$. Find the eigenvalues of A_ϵ and compare with the eigenvalues of A_0 when ϵ is small. For example, let $n = 10$ and take $\epsilon = 10^{-10}$.

   (b) Compute the singular values of A_ϵ and check that 5(b) really does hold in this case.

6. In this problem, let $\langle A, B\rangle_F = \mathrm{tr}(B^H A)$ denote the Frobenius (or Hilbert-Schmidt) inner product on $\mathbf{C}^{n\times n}$, and $\|A\|_F = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$ the associated norm. Consider the problem of minimizing $\|A - B\|_F$ for a given $A \in \mathbf{C}^{n\times n}$, where B ranges over all scalar multiples α of unitary matrices $W \in \mathbf{C}^{n\times n}$.

   (a) Show that if $\alpha \in \mathbf{C}$ and $W \in \mathbf{C}^{n\times n}$ is unitary, then $\|A - \alpha W\|_F^2 \geq \|A\|_F^2 - \frac{1}{n}|\langle A, W\rangle|^2$.

   (b) Show that $\sup_{W \text{ unitary}} |\langle A, W\rangle| = \sum_{i=1}^n \sigma_i$, where the sup is taken over all unitary matrices $W \in \mathbf{C}^{n\times n}$, and $\sigma_1 \geq \ldots \geq \sigma_n$ are the singular values of A.

   (c) Identify α, W such that $B = \alpha W$ solves the minimization problem, and identify the minimum value of $\|A - B\|_F$.

7. (QR Factorization by Householder Transformations.) For simplicity, we consider only the real case. A matrix $Q \in \mathbf{R}^{n\times n}$ of the form $Q = I - 2yy^T/(y^T y)$, where $y \in \mathbf{R}^n$ is called a *Householder reflection.* Applying Q to a vector $x \in \mathbf{R}^n$ reflects that vector in the hyperplane perpendicular to y.

   (a) Show that Q is symmetric and orthogonal; i.e., $Q = Q^T = Q^{-1}$.

   (b) Show that $Qy = -y$ and $Qz = z$ if $z \perp y$. (Hence the geometrical description of Q as a reflection.)

   (c) Given $v \in \mathbf{R}^n$ with $v \neq 0$, show that the Householder reflection with $y = v \pm \|v\|_2\, e_1$ maps v into a multiple of e_1, the first unit vector.

(d) Suppose $A \in \mathbf{R}^{m\times n}$ (with $m \geq n$) has full rank (i.e., $\text{rank}(A) = n$). Use (c) to find a Householder reflection $Q_1 \in \mathbf{R}^{m\times m}$ for which the first column of $Q_1 A$ is a multiple of e_1:

$$Q_1 A = \begin{pmatrix} * & * & \cdots & * \\ 0 & \hat{a}_{22} & \cdots & \hat{a}_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & \hat{a}_{m2} & \cdots & \hat{a}_{mn} \end{pmatrix}$$

Now consider the bottom $m-1$ by $n-1$ submatrix and find a Householder reflection in $\mathbf{R}^{(m-1)\times(m-1)}$ for which the first column of this matrix is mapped to a multiple of e_1. Explain how you can form a second orthogonal matrix $Q_2 \in \mathbf{R}^{m\times m}$ so that $Q_2 Q_1 A$ has zeros below the diagonal in the first two columns. Proceed inductively to show that there are orthogonal transformations $Q_1, \ldots, Q_n$ (or $Q_1, \ldots, Q_{n-1}$ if $m = n$) such that $Q_n Q_{n-1} \cdots Q_1 A = R$ has zeros below the diagonal in columns 1 through n; i.e., it is an m by n upper triangular matrix. Conclude that if $Q = (Q_n \cdots Q_1)^T = Q_1^T \cdots Q_n^T$, then Q is orthogonal and $A = QR$.

8. (a) Let $L \in \mathcal{L}(V)$ where $\dim(V) < \infty$, and let λ be an eigenvalue of L. Show that the geometric multiplicity of λ is equal to its algebraic multiplicity if and only if λ is a simple pole of the resolvent of L.

   (b) Let V be a finite dimensional inner product space and $L \in \mathcal{L}(V)$ be normal. Show that $\|R(\zeta)\| = 1/d(\zeta, \sigma(L))$, where $\|\cdot\|$ is the operator norm on $\mathcal{L}(V)$ induced by the inner product on V and $d(\zeta, \sigma(L)) = \min_{\lambda\in\sigma(L)} |\zeta - \lambda|$.