Assignment 6. Due **Fri., Nov. 14**.

Reading: Class Notes, pp. 33–40
Chapter 5 in Horn and Johnson.

1. Let $C^1([a,b])$ be the space of continuous functions on $[a,b]$ whose derivative (one-sided derivative at the endpoints) exists and is continuous on $[a,b]$.

   (a) Suppose $u \in C([a,b]) \cap C^1((a,b))$ and u' can be extended continuously to $[a,b]$. Show that $u \in C^1([a,b])$.

   (b) Show that $\|u\| = \sup_{x\in[a,b]} |u(x)| + \sup_{x\in[a,b]} |u'(x)|$ is a norm on $C^1([a,b])$ which makes $C^1([a,b])$ into a Banach space. [Hint for completeness: if $\{u_n\} \subset C^1([a,b])$ satisfies $u_n \to u$ uniformly and $u_n' \to v$ uniformly, take limits in the equation $u_n(x) - u_n(a) = \int_a^x u_n'(s)\,ds$ to show that $u \in C^1([a,b])$ and $u' = v$.]

2. If $0 < \alpha \leq 1$, a function $u \in C([a,b])$ is said to satisfy a Hölder condition of order α (or to be Hölder continuous of order α) if

$$\sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty.$$

   Denote by $\wedge^\alpha([a,b])$ this set of functions.

   (a) Show that

$$\|u\|_\alpha = \sup_{x\in[a,b]} |u(x)| + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

   is a norm which makes $\wedge^\alpha$ into a Banach space.

   (b) Show that $C^1([a,b]) \subset \wedge^\alpha([a,b])$ and that the inclusion map is continuous with respect to the norms defined above (i.e., the norm on $C^1([a,b])$ defined in problem 1 and that on $\wedge^\alpha([a,b])$ defined in 2a).

3. Prove that the dual norm to the ℓ^2 norm on $\mathbf{F}^n$ is again the ℓ^2 norm, and show that the ℓ^2 norm is the only norm on $\mathbf{F}^n$ with this property.