Assignment 5. Due **Fri., Nov. 7**.

Reading: Class Notes, pp. 27–37
Chapter 5 in Horn and Johnson.

1. (The parallelogram law is a necessary and sufficient condition for a norm to be induced by an inner product.)

   (a) Show that if a norm $\|\cdot\|$ on a vector space V comes from an inner product, then $\|\cdot\|$ satisfies the parallelogram law:

   $$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

   [Remark: The converse is true as well: If $\|\cdot\|$ is a norm satisfying the parallelogram law, then it is induced by an inner product. If you want a challenge, try to prove this. Start by using the polarization identity to define $\langle x, y\rangle$:

   $$\text{If } \mathbf{F} = \mathbf{R}, \quad \langle x, y\rangle = \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$
   $$\text{If } \mathbf{F} = \mathbf{C}, \quad \langle x, y\rangle = \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \imath\|x+\imath y\|^2 - \imath\|x-\imath y\|^2\right),$$

   and then use the parallelogram law to show that this really defines an inner product which induces $\|\cdot\|$.]

   (b) Is $C([a,b])$ with norm $\|u\| = \sup_{x\in[a,b]} |u(x)|$ an inner product space?

   (c) Show that $\ell^2 \subset \mathbf{F}^\infty$ is the only inner product space among the ℓ^p spaces.

2. A sequence $\{v_n\}$ in an inner product space $(V, \langle\cdot,\cdot\rangle)$ is said to *converge weakly* to v if $(\forall w \in V)\ \langle v_n, w\rangle \to \langle v, w\rangle$, and it is said to *converge strongly* to v if $\|v_n - v\| \to 0$.

   (a) Show that if $\dim(V) < \infty$ and $v_n \to v$ weakly, then $v_n \to v$ strongly.

   (b) Show that part (a) fails for $V = \ell^2$.

   (c) Show that in any inner product space $(V, \langle\cdot,\cdot\rangle)$, if $v_n \to v$ strongly, then $v_n \to v$ weakly.

3. **(a)** Show that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$, where $\|x\|_p$ denotes the ℓ^p-norm of $x \in \mathbf{C}^n$.

   (b) Show that $\|u\|_\infty = \lim_{p\to\infty} \|u\|_p$, where $\|u\|_p$ denotes the L^p-norm of $u \in C([a,b])$.

4. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V. Suppose that all sequences $\{v_n\} \subset V$ which satisfy $\|v_n\|_1 \to 0$ also satisfy $\|v_n\|_2 \to 0$, and vice-versa. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.