Assignment 2. Due Fri., Oct. 10.

Reading: Class Notes, pp. 1–21
Continue your review of linear algebra, including determinants.

1. Define subspaces $W_1, W_2 \subset C([0,1])$ by $W_1 = \{f : \int_0^1 f(x)\,dx = 0\}$ and $W_2 = \{f : f(x) = c$ for some $c \in \mathbf{C}$ and all $x \in [0,1]\}$. Show that $C([0,1]) = W_1 \oplus W_2$ and derive explicit formulas for the projection operators P_1 and P_2 onto W_1 and W_2.

2. Let V and W be finite dimensional vector spaces of dimension n and m, respectively. Let $L \in \mathcal{B}(V,W)$ and suppose $\mathrm{rank}(L) = 1$. Show that for any choice of bases for V and W the matrix of L is of the form $\mathbf{a}\mathbf{b}^T$, where $\mathbf{a} = (a_1, \ldots, a_m)^T$ and $\mathbf{b} = (b_1, \ldots, b_n)^T$.

3. Consider the linear transformation $d/dt : p(t) \to p'(t)$ acting on the vector space $\mathcal{P}_3(\mathbf{R})$ consisting of all (real-valued) polynomials of degree 3 or less.

   (a) Show that d/dt is nilpotent.

   (b) Write down the matrix of this transformation in the basis $\{1, t, t^2, t^3\}$.

   (c) Write down a basis in which d/dt acts as a 4×4 shift.

4. Let V be an n-dimensional vector space and L be a nilpotent linear transformation on V. For $i = 1, \ldots, n$, let $k_i = \dim(\mathcal{N}(L^i))$. Refer to the proof on pp. 17–19 of the Notes to answer the following questions:

   (a) Show how it follows from the proof that

   $$0 < k_1 < k_2 < \ldots < k_r = k_{r+1} = \ldots = n;$$

   hence $r \leq n$.

   (b) Express $\ell_1, \ldots, \ell_r$ in terms of $k_1, \ldots, k_n$.

5. (In this problem, take $\mathbf{F} = \mathbf{R}$.) Let $A \in \mathbf{R}^{n \times n}$. One can regard the entries a_{ij} as independent variables and $\det(A)$ as a function of these n^2 variables.

   (a) Show that $\partial/\partial a_{ij}(\det(A)) = \hat{A}_{ij}$, where $\hat{A}_{ij}$ is the (i,j) cofactor of A; i.e., $\hat{A}_{ij} = (-1)^{i+j}\det(A[i|j])$, where $A[i|j]$ is the $n-1$ by $n-1$ submatrix of A obtained by removing the ith row and jth column of A.

   (b) Suppose $A(t)$ is an $\mathbf{R}^{n \times n}$-valued function of $t \in \mathbf{R}$ whose entries $a_{ij}(t)$ are differentiable functions of t. Show that

   $$\frac{d}{dt}\left(\det A(t)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{A}_{ij}(t)\frac{d}{dt}\left(a_{ij}(t)\right).$$

(c) Suppose in addition that $A(0) = I$. Show that

$$\frac{d}{dt}\left(\det A(t)\right)\big|_{t=0} = \operatorname{tr}\left(\frac{dA}{dt}(0)\right).$$

(d) Suppose $A(t)$ is as in part (b), and suppose $\det A(t) \neq 0$ for all $t \in \mathbf{R}$. Show that

$$\frac{d}{dt}\log\left(\det A(t)\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}(A^{-1})_{ji}\,\frac{d}{dt}\left(a_{ij}(t)\right),$$

where $(A^{-1})_{ji}$ is the (j,i)-element of $(A(t))^{-1}$.