Practice Problems for Midterm (Fri., Feb. 29).

1. Let N be the random variable which counts the number of books a library patron checks out. The library limits the maximum number of books one can check out to 3. Suppose the probability distribution of N is determined by

1	2	3
$\frac{2}{10}$	$\frac{5}{10}$	$\frac{3}{10}$

   (a) How many books on average does a patron take out?

   (b) The library has data on the number of patrons visiting each day and the total number of books borrowed each day for the past year. About 1000 patrons visit the library each day. What should the histogram of the daily average number of books per customer look like? Sketch it, labeling important points, and explain your reasoning.

2. Suppose we have a uniform random number generator, such as `rand` in MATLAB, that generates random numbers between 0 and 1 from a uniform distribution. We wish to generate random numbers from a distribution X, where

$$\text{prob}(X \leq a) = \sqrt{a}, \quad 0 \leq a \leq 1.$$

   How could we use the uniform random number generator to obtain random numbers with this distribution?

3. Suppose a real symmetric matrix A has eigenvalues -4, -2, 1, 3, and 5. Assume that the initial vector in the following algorithms has nonzero components in the direction of each eigenvector.

   (a) To which eigenvalue (if any) will the unshifted power method converge?

   (b) Derive an expression showing the *rate of convergence* of the power method.

   (c) To which eigenvalue (if any) will unshifted inverse iteration converge?

   (d) Suppose you wish to compute the eigenvector corresponding to the eigenvalue 3. What range of shifts could you use with inverse iteration to make it converge to that eigenvector?

4. Consider the following matrix:

$$A = \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}.$$

(a) Compute the QR factorization of A. Explicitly write down the orthogonal matrix Q and the upper triangular matrix R, and show how you found them.

(b) Write down the first step of the QR algorithm for A; that is, compute the first matrix A_1 that is similar to A in the QR algorithm.

(c) Explain what information Gerschgorin's theorem provides about the eigenvalues of A. Draw a graph showing the region(s) in which the eigenvalues lie.

5. Consider the shifted QR algorithm. Suppose $A_k - s_k I = Q_k R_k$ and $A_{k+1} = R_k Q_k + s_k I$. Show that A_{k+1} is orthogonally similar to A_k. Be sure to justify all steps.

6. Consider a derivative approximation of the form

$$f'(x) \approx Af(x) + Bf(x-h) + Cf(x-2h).$$

(a) Use Taylor's theorem to determine coefficients A, B, and C for which this approximation is second order accurate. Justify your answer.

(b) Suppose you use your formula from (a) to generate an approximation $A_{.01}$ with $h = .01$ and another approximation $A_{.005}$ with $h = .005$. What linear combination of these two results would likely give a more accurate approximation to f' and why? [Hint: Richardson extrapolation.]

7. Consider the predator-prey equations ($r(t)$ is the prey population, $s(t)$ is the predator population):

$$\begin{aligned} r' &= (2-s)r \\ s' &= (r-2)s \end{aligned}$$

Starting with $r_0 = 2$ and $s_0 = 1$, determine r_1 and s_1:

(a) Using Euler's method with stepsize $h = 0.1$. [Recall that Euler's method for solving $\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t))$ is: $\mathbf{y_{k+1}} = \mathbf{y_k} + h\mathbf{f}(t_k, \mathbf{y_k})$, $k = 0, 1, \ldots$.]

(b) Using Heun's method with stepsize $h = 0.1$. [Recall that Heun's method for solving $\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t))$ is: $\mathbf{y_{k+1}} = \mathbf{y_k} + (h/2)[\mathbf{F_1} + \mathbf{F_2}]$, where $\mathbf{F_1} = \mathbf{f}(t_k, \mathbf{y_k})$ and $\mathbf{F_2} = \mathbf{f}(t_{k+1}, \mathbf{y_k} + h\mathbf{f}(t_k, \mathbf{y_k})$.]

8. Consider the one-step method

$$y_{k+1} = y_k + h[\theta f(t_k, y_k) + (1-\theta) f(t_{k+1}, y_{k+1})],$$

where $\theta \in [0, 1]$ is given. Note that this method is *explicit* if $\theta = 1$ and otherwise it is *implicit*. Show that the local truncation error,

$$\frac{y(t_{k+1}) - y(t_k)}{h} - [\theta f(t_k, y(t_k)) + (1-\theta) f(t_{k+1}, y(t_{k+1}))],$$

where y is a true solution of the equation $y' = f(t, y)$, is $O(h^2)$ if $\theta = 1/2$ and otherwise is $O(h)$.