Math 498 | Chartier/Greenbaum
Homework #6 – Solutions | Winter 2008

NUMERICAL DIFFERENTIATION

1. From (Chapter 9) page 181 of the Course notes

Exercise 1 The MATLAB program `ch9Exercise1.m` posted on the course web page produces the following output:

```
     h        Diff. Quotient        Error
=============================================
1.00e-001  -4.99583472e-001     4.16527803e-004
1.00e-002  -4.99995833e-001     4.16665263e-006
1.00e-003  -4.99999958e-001     4.16744967e-008
1.00e-004  -4.99999997e-001     3.03873549e-009
1.00e-005  -5.00000596e-001     5.96481698e-007
1.00e-006  -4.99933428e-001     6.65720113e-005
1.00e-007  -4.94049246e-001     5.95075404e-003
1.00e-008  -1.11022302e+000     6.10223025e-001
1.00e-009   5.55111512e+001     5.60111512e+001
1.00e-010   0.00000000e+000     5.00000000e-001
1.00e-011   0.00000000e+000     5.00000000e-001
1.00e-012   0.00000000e+000     5.00000000e-001
1.00e-013   5.55111512e+009     5.55111512e+009
1.00e-014  -5.55111512e+011     5.55111512e+011
1.00e-015   0.00000000e+000     5.00000000e-001
1.00e-016  -5.55111512e+015     5.55111512e+015
```

As expected that the smallest error occurs at $\varepsilon^{1/4}$ as discussed in lecture and the course notes.

Exercise 2 Notice that formula (7.3) the centered difference formula has the same form as the example given in class on Richardson's extrapolation as applied to an integration problem. Therefore, the same procedure of canceling terms will produce the same extrapolation formulas. See the Excel document on the course web page for an example of how to perform Richardson's Extrapolation with Excel.

Exercise 3 First note that

$$\begin{aligned} f(x+h) &= f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{3!}f'''(c_1) \\ f(x+2h) &= f(x)+2hf'(x)+2h^2f''(x)+\frac{4h^3}{3}f'''(c_2) \end{aligned}$$

Therefore, $4f(x+h)-f(x+2h) = 3f(x)+2hf'(x)+\frac{4h^3}{6}(f'''(c_1)-2f'''(c_2))$. So, we get:

$$f'(x) = \frac{1}{2h}\left[-3f(x)+4f(x+h)-f(x+2h)\right]+O(h^2).$$

So, the method is $O(h^2)$.

Exercise 4 We have

$$
\begin{aligned}
f(x+h) &= f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{3!}f'''(c_1) \\
f(x+2h) &= f(x)+2hf'(x)+2h^2f''(x)+\frac{4h^3}{3}f'''(c_2).
\end{aligned}
$$

Therefore, $f(x+2h)-2f(x+h) = -f(x)+h^2f''(x)+\frac{h^3}{6}(8f'''(c_2)-2f'''(c_1))$. Therefore,

$$
f''(x) = \frac{f(x)-2f(x+h)+f(x+2h)}{h^2}+O(h).
$$

This method is $O(h)$ with $A = 1/h^2, B = -2/h^2$ and $C = 1/h^2$.

2. First, we compute the Taylor expansions of $f(x+h)$ and $f(x-h)$:

$$
\begin{aligned}
f(x+h) &= f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{(4)}(x)+\frac{h^5}{5!}f^{(5)}(c_1) \\
f(x-h) &= f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{(4)}(x)-\frac{h^5}{5!}f^{(5)}(c_2)
\end{aligned}
$$

Subtracting these two expansions:

$$
f(x+h)-f(x-h) = 2hf'(x)+\frac{h^3}{3}f'''(x)+\frac{h^5}{5!}(f^{(5)}(c_1)+f^{(5)}(c_2)).
$$

Next, compute the Taylor series of $f'(x+h)$ and $f'(x-h)$:

$$
\begin{aligned}
f'(x+h) &= f'(x)+hf''(x)+\frac{h^2}{2}f'''(x)+\frac{h^3}{3!}f^{(4)}(x)+\frac{h^4}{4!}f^{(5)}(c_3) \\
f'(x-h) &= f'(x)-hf''(x)+\frac{h^2}{2}f'''(x)-\frac{h^3}{3!}f^{(4)}(x)+\frac{h^4}{4!}f^{(5)}(c_4)
\end{aligned}
$$

Add these two expansions together:

$$
f'(x-h)+f'(x+h) = 2f'(x)+h^2f'''(x)+\frac{h^4}{4!}(f^{(5)}(c_3)+f^{(5)}(c_4)).
$$

Multiply this last equation by $\frac{h}{3}$ and subtracting it from $f(x+h)-f(x-h)$ to cancel the leading order error term in each equation (the $f'''(x)$ term). The $f^{(5)}$ term will remain and give the error for our differentiation formula. In particular, we get:

$$
(f(x+h)-f(x-h))-\frac{h}{3}(f'(x-h)+f'(x+h)) \quad = \quad \frac{4h}{3}f'(x)+O(h^5)
$$

Solving for $f'(x_0)$ gives our $O(h^4)$ approximation:

$$
f'(x_0) = -\frac{f'(x_0+h)+f'(x_0-h)}{4}+3\frac{f'(x_0+h)-f'(x_0-h)}{4h}+O(h^4).
$$

Therefore, $\alpha = -\frac{1}{2}$ and $\beta = \frac{3}{2}$.