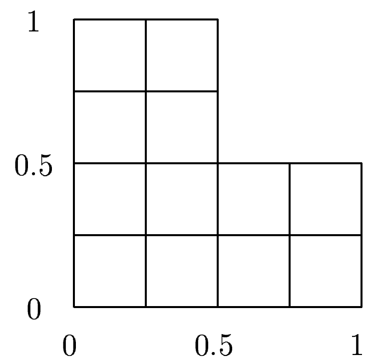
Practice Problems for Midterm (Wed., May 2, 2007).

1. Consider Laplace's equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ on the L-shaped domain pictured below:



with $u = 1$ on the boundary.

Write down a second-order accurate set of difference equations for this problem, showing how you number the nodes and what coefficient matrix and right-hand side vector you obtain.

2. Consider the two-point boundary value problem

$$\frac{d}{dx}\left(e^x \frac{du}{dx}\right) = f(x), \quad 0 \le x \le 1$$

$$u(0) = 1, \quad u'(1) = 1.$$

(a) Letting $h = 1/4$, write down a second-order accurate set of difference equations for this problem.

(b) Suppose you wish to test a code for solving this problem by choosing $f(x)$ so that the true solution will be $u(x) = e^{x^2 - x}$, and then comparing your computed solution to the true solution. What must $f(x)$ be?

3. Consider the two-point boundary value problem

$$-u''(x) + (1 + x^2)u(x) = f(x), \quad 0 \le x \le 1$$

$$u(0) = u(1) = 0.$$

Suppose we divide the interval from 0 to 1 into n equal subintervals each of width $h = 1/n$ and replace the differential equation by the following set of difference equations:

$$\frac{2u_i - u_{i+1} - u_{i-1}}{h^2} + (1 + x_i^2)u_i = f(x_i), \quad i = 1, \ldots, n-1.$$

(a) Show that the local truncation error for this method is $O(h^2)$.

(b) Write the above equations in matrix form, $A\mathbf{u} = \mathbf{f}$, where $\mathbf{u} = (u_1, \ldots, u_{n-1})^T$ is the vector of unknowns. Is the matrix A *tridiagonal*? Is it *symmetric*? Is it *Toeplitz*?

(c) Show that the matrix A in (b) is nonsingular. Give upper and lower bounds on its eigenvalues using Gerschgorin's theorem.

(d) Give a bound on the 2-norm of A^{-1}. What can you conclude about the L_2-norm of the *global error* (the difference between the computed values u_i and the true solution values $u(x_i)$)?

4. Consider a derivative approximation of the form

$$f'(x) \approx Af(x) + Bf(x-h) + Cf(x-2h).$$

Use Taylor's theorem to determine coefficients A, B, and C for which this approximation is second order accurate. Justify your answer.

5. Suppose you approximate $f'(x)$ as in the previous problem, using $h = 0.1$ and $h = 0.05$. Explain what linear combination of these two results would likely give a more accurate approximation to $f'(x)$ and why. (Hint: Use Richardson extrapolation.)

6. Suppose you have measured values for $f(1)$, $f(1.1)$, and $f(1.2)$, say,

$$f(1) = 2.01, \quad f(1.1) = 2.02, \quad f(1.2) = 2.04.$$

You wish to approximate $f''(1.1)$ using the formula

$$f''(x) \approx \frac{f(x+h) + f(x-h) - 2f(x)}{h^2},$$

where $x = 1.1$ and $h = .1$. The measurements are accurate to the two decimal places printed; that is, the true value of f lies within $\pm.005$ of the measured value. About how accurate an approximation to f'' would you expect to obtain and why?