Assignment 2.

Due Friday, Apr. 18.

Objectives: To study collocation and finite element methods.

(1) Consider the two-point boundary value problem

$$u''(x) = u(x) + x^2, \quad 0 \le x \le 1$$

$$u(0) = u(1) = 0.$$

(a) Consider the basis functions $\phi_j(x) = \sin(j\pi x)$, $j = 1, 2, 3$, and the collocation points $x_i = i/3$, $i = 1, 2, 3$. Suppose $u(x)$ is to be approximated by a linear combination of these basis functions: $u(x) \approx \sum_{j=1}^{3} c_j \phi_j(x)$, and suppose the coefficients are to be chosen so that the differential equation holds at the collocation points. Write down the system of linear equations that you would need to solve to determine c_1, c_2, and c_3.

(b) Repeat part (a) with $\phi_j(x) = x^j(1-x)$, $j = 1, 2, 3$.

(2) Show that the functions $\sin(k\pi x)$ are mutually orthogonal on the interval $[0, 1]$; that is,

$$\int_0^1 \sin(k\pi x)\sin(j\pi x)\, dx = 0, \quad \text{if } j \ne k.$$

(3) Use the Galerkin method with continuous piecewise linear basis functions to solve problem 3 on Assignment 1, but with homogeneous Dirichlet boundary conditions. The new problem is then:

$$\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f(x), \quad 0 \le x \le 1,$$

$$u(0) = 0, \quad u(1) = 0.$$

(a) Derive the matrix equation that you will need to solve for this problem.

(b) Write a MATLAB code to solve this set of equations. You can test your code on a problem where you know the solution by choosing a function $u(x)$ that satisfies the boundary conditions and determining what $f(x)$ must be in order for $u(x)$ to solve the problem. Try $u(x) = x(1-x)$. Then $f(x) = -2(3x^2 - x + 1)$.

(c) Try several different values for the mesh size h. Based on your results, what would you say is the order of accuracy of the Galerkin method with continuous piecewise linear basis functions?