Math 464, Autumn 2006

Practice Problems for Final (Wed., Dec. 13, 8:30–10:20).

Final will cover chapters 1 through 6.4, with special emphasis on material since the midterm (i.e., chs. 5 and sections 1 through 4 of chapter 6). This sheet contains practice problems only for chs. 5 and 6 (except the first problem, which is a followup from the midterm). Refer to earlier practice sheet for problems from chs. 1–4.

1. **(a)** On the midterm we considered the problem of evaluating $f(x) = 1 - \sqrt{1-x}$ and showed that while the problem was well-conditioned for x near 0, one would get an inaccurate result by applying this formula in a straightforward way. For example, if $|x| < 10^{-16}$, then $1-x$ would be rounded to 1, taking $\sqrt{1}$ would give 1, and then subtracting this from 1 would give an answer of 0, which does not have high relative accuracy. Write down an algorithm that will give high relative accuracy. [Hint: One possible approach: You can evaluate $1+\sqrt{1-x}$ to high relative accuracy. We have the product formula $(1-\sqrt{1-x})(1+\sqrt{1-x}) = x$.]

   (b) Suppose we wish to solve the quadratic equation

   $$x^2 + bx + c = 0$$

   for x, where $b > 0$ and $|c| << b^2$. The formulas for the two roots are

   $$x_+ = \frac{-b+\sqrt{b^2-4c}}{2}, \quad x_- = \frac{-b-\sqrt{b^2-4c}}{2}$$

   Write down an algorithm that can be used to evaluate both x_+ and x_- to high relative accuracy.

2. The following fragment of MATLAB code does Gaussian elimination without pivoting on an n by n matrix A:

```
for k=1:n-1,                  % Use row k to eliminate entries in column k
                              % of rows k+1 through n.
                              %
                              %   Here you should build in partial pivoting
                              %
  for i=k+1:n,
    mult = A(i,k)/A(k,k);               % Subtract mult times row k from row i
    for j=k:n,                          % in order to zero out A(i,k)
      A(i,j) = A(i,j) - mult*A(k,j);
    end;
  end;
end;
```

(a) Write down the code you would insert to implement partial pivoting. (If you are not sure about the MATLAB commands, you may write your code in C or in some pseudo-MATLAB form, as long as it is clear *exactly* what you are doing.)

(b) Suppose A is *tridiagonal* and pivoting is not required. Show how you could modify the above code to solve this problem efficiently, and count the number of operations performed in your modified code.

3. (a) Compute the 2-norm and the ∞-norm of the vector:

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

(b) What is the ∞-norm ($\max_{\|v\|_\infty=1} \|Av\|_\infty$) of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6.1 \end{pmatrix}?$$

(c) Determine the *condition number* of A in the ∞-norm?

4. (a) Let x be the exact solution to the linear system $Ax = b$, and let $\hat{x}$ be the exact solution to the linear system $A\hat{x} = \hat{b}$, where A is a nonsingular matrix. Derive a bound on the relative error, $\|\hat{x} - x\|/\|x\|$, in terms of the relative change in b, $\|\hat{b} - b\|/\|b\|$.

(b) What does it mean for an algorithm to be *backward stable*? If a backward stable algorithm is used to solve a linear system $Ax = b$ on a machine with unit roundoff ϵ, approximately how large will the relative error, $\|\hat{x} - x\|/\|x\|$, be?

5. Factor the following matrix in the form QR, where Q is a 3 by 2 matrix with orthonormal columns and R is a 2 by 2 upper triangular matrix:

$$A = \begin{pmatrix} 0 & -4 \\ 0 & 0 \\ -5 & -2 \end{pmatrix}.$$

Use your QR factorization to solve the least squares problem $Ax \approx b$, where

$$b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

6. Consider the following set of data:

x	y
1	1
2	2
3	4

(a) Find the straight line that best fits this data in a least squares sense. Show how you obtained your answer. Also plot the data points and the straight line that you computed.

(b) Write down the Lagrange form of a quadratic polynomial that exactly fits the data.

(c) Write down the Newton form of a quadratic polynomial that exactly fits the data.

(d) Assume that this data comes from a function f whose derivatives are all continuous. Write an expression for the difference between $f(x)$ and your quadratic interpolant at an arbitrary point x. Suppose that all of the derivatives of f are bounded in absolute value by 3; that is, $|f^{(n)}(x)| \leq 3$ for all x and for $n = 1, 2, \ldots$. Give a bound on the difference between $f(1.5)$ and the value of your quadratic at $x = 1.5$.

7. Compare the efficiency of the divided difference algorithm for computing the coefficients in the nth degree Newton interpolant of a function f to that of solving a triangular system for these coefficients.