Practice Problems for Midterm (Wed., Nov. 8).

Midterm will cover through section 5.2.1.

1. It takes about 1 second to solve a 1000 by 1000 linear system on a certain computer using Gaussian elimination. About how long would it take to solve a 5000 by 5000 linear system? Suppose the 5000 by 5000 matrix has already been factored in the form LU, where L is unit lower triangular and U is upper triangular. How can you use the L and U factors to solve the linear system $Ax = b$ more efficiently, and about how long would this take?

2. The following fragment of MATLAB code does just *one* step of Gaussian elimination on an n by n matrix A, using the first row to eliminate elements in the first column of rows 2 through n:

```
for i=2:n,
  mult =                                  % Fill in the value for "mult".
  for j=1:n,                                % We could replace this inner "for"
    A(i,j) = A(i,j) - mult*A(1,j);          % loop by the single statement
  end;                                      % A(i,:) = A(i,:) - mult*A(1,:);
end;
```

   In other words, it transforms A as follows:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & \tilde{a}_{n2} & \dots & \tilde{a}_{nn} \end{pmatrix}.$$

   (a) Fill in the proper expression for `mult`.

   (b) How many floating point operations (additions, subtractions, multiplications, and divisions) are performed in this first stage of the elimination algorithm? Show your work in deriving this result.

   (c) Eliminating elements below the diagonal in column one corresponds to multiplying A on the left by a certain lower triangular matrix. Write down this lower triangular matrix.

3. Determine the absolute and relative condition numbers for the function $f(x) = \sin(x)$. What is the limit of the relative condition number as x approaches 0? Determine relative condition numbers for the functions $\ln x$, $x^{-1}e^x$, and $\arcsin(x)$. Where are these functions ill-conditioned? Let $g(\alpha)$ be the positive root of $f(x) \equiv x^2 - \alpha$, where $\alpha \geq 0$. Determine the absolute and relative condition number for g.

4. Using IEEE single precision and round to nearest, what numbers are rounded to zero, and what numbers are rounded to the smallest subnormal number?

   Using IEEE single precision and round to nearest, what is the largest floating point number a for which $1 \oplus a = 1$?

5. If a is a normalized floating point number, is the floating point product $1 \otimes a$ equal to a? If a is a nonzero normalized floating point number, is the floating point quotient $a \oslash a$ equal to 1? If a and b are normalized floating point numbers and the floating point difference $a \ominus b$ is zero, does this imply that $a = b$?

6. What are the IEEE single precision representations for $1/10$, using each of the four rounding modes? What are they for $1 + 2^{-25}$ and for 2^{130}?

7. Show how Newton's method can be derived from a truncated Taylor series expansion.

8. Give a geometric derivation of Newton's method.

9. Express Newton's method for finding a root of a function f as a fixed point iteration for finding a fixed point of a function φ; i.e., what is the function φ, and why does it have a fixed point where f has a root.

10. Show that the equation $x^2 - 5 = 0$ is equivalent to the fixed-point problem $x = x + c(x^2 - 5) \equiv \varphi(x)$, where c is any nonzero constant. Determine the values of c for which the fixed-point iteration $x_{n+1} = x_n + c(x_n^2 - 5)$ is guaranteed to converge to $x_* = \sqrt{5}$, starting with $x_0 = 1$.

11. Write down the first two iterates in Newton's method for finding a root of $f(x) = x^2 - 2$, starting with $x_0 = 1$. Do the same for the secant method, starting with $x_0 = 1$ and $x_1 = 2$. Do the same for the bisection method, using the initial interval $[1, 2]$. How many steps would be required by the bisection method to reduce the interval size to 10^{-6}? If the error at step k of Newton's method is 10^{-3}, approximately what size error would you expect at step $k + 1$? Explain your answer. Same question for the secant method.

12. Suppose Newton's method is used to find a root of $f(x) = x - \sin(x)$, which has a root at $x = 0$. What convergence rate would you expect and why? Could the bisection method be used to find a root of this function? Why or why not?