Practice Problems on Convergence of Sequences and Series, Pointwise vs. Uniform Convergence

1. Test each of the following series for convergence.

   (a) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \)
   
   (b) \( \sum_{n=1}^{\infty} \frac{n!}{3^n} \)
   
   (c) \( \sum_{n=1}^{\infty} \frac{n^{n+1/n}}{(n+1/n)^n} \)
   
   (d) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{2^n} \)
   
   (e) \( \sum_{n=2}^{\infty} \frac{1}{(3n)^n} \)

2. If \( \sum a_n \) and \( \sum b_n \) are absolutely convergent, show that \( \sum (a_n + b_n) \) is also. If \( \sum a_n \) is absolutely convergent, show that \( \sum a_n^2 \) and \( \sum a_n/(1 + a_n) \) (where \( a_n \neq -1 \) for any \( n \)) are as well.

3. Discuss the convergence of the following series. State when the series converges absolutely and when it converges conditionally.

   (a) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/n}} \)
   
   (b) \( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots \)

4. Show that the sequence of functions \( f_n(x) = \frac{x^{2n}}{1+x^2} \) converges pointwise but not uniformly on the entire real line.

5. Show that while the series \( \sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n^2} \) converges uniformly on the entire real line, the series cannot be differentiated term by term on any open interval.

6. Use the fact that \( \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots \) if \( -1 < t < 1 \) and the fact that \( \ln(1+x) = \int_0^x \frac{1}{1+t} \, dt \) to derive the series expansion

   \[
   \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad |x| < 1.
   \]

7. Find the interval of convergence of the following power series.

   (a) \( \sum_{n=1}^{\infty} n! (x - 3)^n \)
   
   (b) \( \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \)