Practice Problems on Convergence of Sequences and Series, Pointwise vs. Uniform Convergence

1. Test each of the following series for convergence.

   (a) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

   (b) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

   (c) $\sum_{n=1}^{\infty} \frac{n^{n+1/n}}{(n+1/n)^n}$

   (d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$

   (e) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

2. If $\sum a_n$ and $\sum b_n$ are absolutely convergent, show that $\sum(a_n + b_n)$ is also. If $\sum a_n$ is absolutely convergent, show that $\sum a_n^2$ and $\sum a_n/(1 + a_n)$ (where $a_n \neq -1$ for any n) are as well.

3. Discuss the convergence of the following series. State when the series converges absolutely and when it converges conditionally.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n-\ln n}$

   (b) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$

4. Show that the sequence of functions $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$ converges pointwise but not uniformly on the entire real line.

5. Show that while the series $\sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n^2}$ converges uniformly on the entire real line, the series cannot be differentiated term by term on any open interval.

6. Use the fact that $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots$ if $-1 < t < 1$ and the fact that $\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$ to derive the series expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad |x| < 1.$$

7. Find the interval of convergence of the following power series.

   (a) $\sum_{n=1}^{\infty} n!(x-3)^n$

   (b) $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$