Practice Problems for Midterm 2

1. Let A and B be sets of real numbers. Define the sets $A+B$ and $A \cdot B$ as follows:

$$A+B = \{x+y : \ x \in A \ \text{ and } \ y \in B\}, \quad A \cdot B = \{xy : \ x \in A \ \text{ and } \ y \in B\}.$$

   (a) If A and B are bounded above, prove that $A+B$ is also bounded above and that $\text{lub}(A+B) = \text{lub}(A) + \text{lub}(B)$.

   (b) If A and B are sets of nonnegative numbers which are bounded above, prove that $A \cdot B$ is bounded above and that $\text{lub}(A \cdot B) = \text{lub}(A) \cdot \text{lub}(B)$. Give an example to show that $A \cdot B$ might have no upper bound if A and B are bounded above but are allowed to contain negative numbers.

2. Let $S = \{\cos(m/n) : \ m, n \in \mathbf{Z}^+\}$. What are $\text{lub}(S)$ and $\text{glb}(S)$? Is S a closed set? Is S a countable set?

3. Let a_1 and b_1 be any two positive numbers with $a_1 < b_1$. Define sequences $\{a_n\}$ and $\{b_n\}$ by

$$a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{1}{2}(a_n + b_n), \quad n = 1, 2, \ldots.$$

   Prove that the sequences converge and have the same limit. [Hint: Show by induction that $\{a_n\}$ is an increasing sequence bounded above by b_1 and that $\{b_n\}$ is a decreasing sequence bounded below by a_1. Conclude that each sequence has a limit. Take limits on both sides of the equation $b_{n+1} = \frac{1}{2}(a_n + b_n)$ to show that the limits are the same.]

4. Show *directly* (i.e., without using the theorem that a convergent sequence is Cauchy) that each of the following sequences is a Cauchy sequence:

   (a) $\left\{\frac{1}{n}\right\}$

   (b) $\left\{\frac{(-1)^n}{n}\right\}$

   (c) $3, 3.3, 3.33, 3.333, \ldots$, where the nth number in the sequence is $\sum_{i=0}^{n-1}(3/10^i)$.

5. Try exercise 5 on p. 83 of the text, exercises 4 and 5 on p. 516-517, and the exercise on p. 523.