Partial Solutions for Practice Problems for Midterm 2

1. Let A and B be sets of real numbers. Define the sets $A+B$ and $A\cdot B$ as follows:

$$A+B=\{x+y:\ x\in A \ \text{ and } \ y\in B\},\quad A\cdot B=\{xy:\ x\in A \ \text{ and } \ y\in B\}.$$

   (a) If A and B are bounded above, prove that $A+B$ is also bounded above and that $\mathrm{lub}(A+B)=\mathrm{lub}(A)+\mathrm{lub}(B)$.

   Let $M_A=\mathrm{lub}(A)$ and $M_B=\mathrm{lub}(B)$. Then if $x\in A$ and $y\in B$, then $x\le M_A$ and $y\le M_B$, so $x+y\le M_A+M_B$; hence M_A+M_B is an upper bound for $A+B$. For any $\epsilon>0$, there is a number $\hat{x}\in A$ such that $\hat{x}>M_A-\epsilon/2$ and there is a number $\hat{y}\in B$ such that $\hat{y}>M_B-\epsilon/2$. Hence $\hat{x}+\hat{y}>(M_A+M_B)-\epsilon$. Therefore $M_A+M_B=\mathrm{lub}(A+B)$.

   (b) If A and B are sets of nonnegative numbers which are bounded above, prove that $A\cdot B$ is bounded above and that $\mathrm{lub}(A\cdot B)=\mathrm{lub}(A)\cdot\mathrm{lub}(B)$. Give an example to show that $A\cdot B$ might have no upper bound if A and B are bounded above but are allowed to contain negative numbers.

   Let $M_A=\mathrm{lub}(A)$ and $M_B=\mathrm{lub}(B)$. Then if $x\in A$ and $y\in B$, then $x\le M_A$ and $y\le M_B$, so (since x and y are nonnegative) $xy\le M_A\cdot M_B$. Therefore $M_A\cdot M_B$ is an upper bound for $A\cdot B$. If $M_A=0$ or $M_B=0$, then $M_A\cdot M_B=0$, and clearly this is the least upper bound for $A\cdot B$, since its elements are nonnegative and, in this case, it would have to consist of the single element 0. Assume, then, that $M_A>0$ and $M_B>0$. For any $\epsilon>0$, there is a number $\hat{x}\in A$ such that $\hat{x}>M_A-\epsilon/(M_A+M_B)$ and a number $\hat{y}\in B$ such that $\hat{y}>M_B-\epsilon/(M_A+M_B)$. Then $\hat{x}\hat{y}>(M_A-\epsilon/(M_A+M_B))\cdot(M_B-\epsilon/(M_A+M_B))=M_A\cdot M_B-\epsilon+\epsilon^2/(M_A+M_B)^2>M_A\cdot M_B-\epsilon$. Therefore $M_A\cdot M_B$ is the least upper bound for $A\cdot B$.
   If $A=\{x:x\le 1\}$ and $B=\{y:y\le 2\}$, since A and B have arbitrarily large (in absolute value) negative numbers and since the product of two negative numbers is positive, $A\cdot B$ has arbitrarily large positive numbers and so is not bounded above by $M_A\cdot M_B=2$.

2. Let $S=\{\cos(m/n):\ m,n\in\mathbf{Z}^+\}$. What are $\mathrm{lub}(S)$ and $\mathrm{glb}(S)$? Is S a closed set? Is S a countable set?

   $\mathrm{lub}(S)=1$ and $\mathrm{glb}(S)=-1$, since these are the lub and glb of $\cos(x)$ for $x\in\mathbf{R}^+$, and any x in $\mathbf{R}^+$ (e.g., $x=\pi$ or $x=2\pi$) can be arbitrarily well approximated by a positive rational number, and then since cosine is a continuous function, the cosine of a rational number sufficiently close to x

will be close to $\cos(x)$. [This is informal. It could be made precise with ϵ's and δ's, but I wasn't suggesting that you do that.]

S is not a closed set, since it does not contain the limit point -1, as $\cos(x) = -1$ if and only if x is an odd integer multiple of π, but such x's are irrational. Also, S does not contain the limit point 1, since neither 0 nor any other integer multiple of 2π can be written in the form m/n where $m, n \in \mathbf{Z}^+$. S is countable, since it can be put in 1-1 correspondence with the positive rationals, which can be put in 1-1 correspondence with the positive integers.

3. Let a_1 and b_1 be any two positive numbers with $a_1 < b_1$. Define sequences $\{a_n\}$ and $\{b_n\}$ by

$$a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{1}{2}(a_n + b_n), \quad n = 1, 2, \ldots.$$

Prove that the sequences converge and have the same limit. [Hint: Show by induction that $\{a_n\}$ is an increasing sequence bounded above by b_1 and that $\{b_n\}$ is a decreasing sequence bounded below by a_1. Conclude that each sequence has a limit. Take limits on both sides of the equation $b_{n+1} = \frac{1}{2}(a_n + b_n)$ to show that the limits are the same.]

First note that since a_1 and b_1 are positive, all a_n and b_n are positive. This can be shown by induction. If a_n and b_n are positive, a_{n+1}, being the square root of a positive number, is positive, and b_{n+1}, being the average of two positive numbers, is positive.

We now show by induction that $a_n < b_n$ for all $n \in \mathbf{Z}^+$. We are told that this holds for $n = 1$. Assuming that $a_n < b_n$, we will show that $a_{n+1} < b_{n+1}$. We have $a_{n+1}^2 = a_n b_n$ and $b_{n+1}^2 = \frac{1}{4}(a_n^2 + 2a_n b_n + b_n^2)$. Hence $b_{n+1}^2 - a_{n+1}^2 = \frac{1}{4}(a_n^2 - 2a_n b_n + b_n^2) = \frac{1}{4}(a_n - b_n)^2 > 0$. [Since $(a_n - b_n)^2 \geq 0$ for any a_n and b_n, we wouldn't even need the induction hypothesis to prove the statement with $>$ replaced by $\geq$, but it certainly follows from the induction hypothesis that $(a_n - b_n)^2 > 0$.] Since a_{n+1} and b_{n+1} are positive numbers, this implies that $a_{n+1} < b_{n+1}$.

It follows that $a_{n+1} = \sqrt{a_n b_n} > \sqrt{a_n a_n} = a_n$, so $\{a_n\}$ is an increasing sequence. Also, $b_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(b_n + b_n) = b_n$, so $\{b_n\}$ is a decreasing sequence. Since each a_n is less than b_n, which is less than or equal to b_1, $\{a_n\}$ is an increasing sequence bounded above by b_1. Since each b_n is greater than a_n, which is greater than or equal to a_1, $\{b_n\}$ is a decreasing sequence bounded below by a_1. Threfore each sequence has a limit, say, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Taking limits on both sides in the equation $b_{n+1} = \frac{1}{2}(a_n + b_n)$, and using the fact that the limit of a sum is the sum of the limits and the limit of a scalar multiple of a sequence is the scalar multiple of the limit, we find $b = \frac{1}{2}(a + b)$, or, $a = b$.

4. Show *directly* (i.e., without using the theorem that a convergent sequence is Cauchy) that each of the following sequences is a Cauchy sequence:

(a) $\left\{\frac{1}{n}\right\}$

$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$. Given $\epsilon > 0$, let $N = 2/\epsilon$. Then $n, m > N$ implies $1/n < \epsilon/2$ and $1/m < \epsilon/2$; hence $1/n + 1/m < \epsilon$. Therefore this sequence is Cauchy.

(b) $\left\{\frac{(-1)^n}{n}\right\}$

Since $\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$, use same argument as in (a).

(c) $3, 3.3, 3.33, 3.333, \ldots$, where the nth number in the sequence is $\sum_{i=0}^{n-1}(3/10^i)$.

Let $m > n$. Then

$$|x_m - x_n| = \left|\sum_{i=n}^{m-1}(3/10^i)\right| = \frac{3}{10^n}\sum_{i=0}^{m-n-1}\frac{1}{10^i} = \frac{3}{10^n}\frac{1-(1/10)^{m-n}}{1-(1/10)} < \frac{3}{10^n}\cdot\frac{10}{9}.$$

Since the right-hand side goes to 0 as $n \to \infty$, given any $\epsilon > 0$, there is an N such that $m > n > N$ implies $|x_m - x_n| < \epsilon$. Therefore this is a Cauchy sequence.