Week 9 Worksheet

1. (a) Find $P_A(t)$ for $A = \begin{bmatrix} 11 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ -16 & 5 & 2 & 0 & 0 \\ 7 & 19 & 13 & 0 & 0 \\ 6 & 82 & -1 & 1 & 3 \end{bmatrix}$ and list the eigenvalues for A.

$$P_A(t) = \det(A - tI) = (11-t)(3-t)^2(2-t)(-t)$$

Eigenvalues: $\lambda = 0, 2, 3, 11$

(b) The matrix A is known as a <u>triangular</u> matrix. Explain how to find $P_A(t)$ and the eigenvalues for these matrices.

If A is triangular, then

$$P_A(t) = (a_{11} - t)(a_{22} - t) \dots (a_{nn} - t)$$

Eigenvalues $\lambda = a_{11}, a_{22}, \dots, a_{nn}$ entries on main diagonal

2. (a) Let $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Calculate D^3.

$$D^3 = \begin{bmatrix} 2^3 & 0 & 0 & 0 \\ 0 & (-1)^3 & 0 & 0 \\ 0 & 0 & 0^3 & 0 \\ 0 & 0 & 0 & 5^3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 125 \end{bmatrix}$$

(b) The matrix D is known as a <u>diagonal</u> matrix. Explain how to calculate D^k for these matrices.

$$D^k = \begin{bmatrix} d_{11}^k & 0 & 0 & \cdots & 0 \\ 0 & d_{22}^k & & & \vdots \\ \vdots & & d_{33}^k & & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & d_{nn}^k \end{bmatrix}$$

(c) Multiplying matrices like D is $\boxed{\text{EASY}}$ when compared to multiplying matrices in general.

3. The matrices P, P^{-1}, and D are given below.

$$P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}$$

(a) Define $A = PDP^{-1}$. Calculate A.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 10 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & -12 \\ 2 & 1 \end{bmatrix}$$

(b) Calculate $P_A(t)$.

$$P_A(t) = \det(A - tI) = \begin{vmatrix} 11-t & -12 \\ 2 & 1-t \end{vmatrix}$$

$$= (11-t)(1-t) + 24 = 11 - 12t + t^2 + 24$$

$$= t^2 - 12t + 35 = (t-5)(t-7)$$

$$\boxed{\text{Eigenvalues: } \lambda = 5, 7}$$

(c) Find bases for each of the eigenspaces of A.

$\lambda = 5$ $\quad A - 5I = \begin{bmatrix} 6 & -12 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$$\mathcal{B}_{E_5(A)} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$\lambda = 7$ $\quad A - 7I = \begin{bmatrix} 4 & -12 \\ 2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$

$$\mathcal{B}_{E_7(A)} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

4. What conclusions can you make in the last problem? What are the entries in D in relation to A? What are the columns of P in relation to A? Is there any relationship between the entries of D and the columns of P?

- Diagonal entries of D are eigenvalues of A.
- Columns of P are the basis vectors for the eigenspaces of A.
- The entries of D & columns of P "match." That is, the i^{th} column of P is a basis vector for the eigenspace for the d_{ii} entry of D.

Goal: Given a matrix A, find P (invertible) and D (diagonal) where $A = PDP^{-1}$. To try this right away, skip the next problem for now and go to the next page.

5. **The Unifying Theorem: The Final Chapter**
Let A be an $n \times n$ matrix. A few of the many parts of the Unifying Theorem are:

$$A \text{ is invertible} \iff \text{null}(A) = \left\{\vec{0}\right\} \iff \det(A) \neq 0$$

With this in mind, try to add something about eigenvalues to the Unifying Theorem. *Hint:* Either consider $E_\lambda(A)$ or $P_A(t)$, and come up with a string of a few "if and only if" statements. You should start with one of the statements above (or its negation) and end with a statement about eigenvalues.

There are several ways to do this. We saw the determinant way in class. Here's another:

One statement from Unifying Thm,

$\text{null}(A) = \{\vec{0}\} \iff A\vec{x} = \vec{0}$ has only the trivial solution

$\iff A\vec{x} = 0\vec{x}$ has only the trivial solution

$\iff \lambda = 0$ is NOT an eigenvalue

6. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 3 \\ 3 & 0 & 5 \end{bmatrix}$. We are told that $P_A(t) = (2-t)^2(5-t)$. Find an invertible matrix P and a *diagonal* matrix D such that $A = PDP^{-1}$. (Hint: Reverse engineer the process in Problem 3.)

Eigenvalues: $\lambda = 2$, $\lambda = 5$

$E_2(A)$: $A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 3 \\ 3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{B}_{E_2(A)} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$E_5(A)$: $A - 5I = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -3 & 3 \\ 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathcal{B}_{E_5(A)} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

If

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \& \quad P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

then $A = PDP^{-1}$

Any order works as long as these are consistent

7. Let $M = \begin{bmatrix} 11 & -12 \\ 2 & 1 \end{bmatrix}$. What is M^{10}? (You should be able to answer this question relatively quickly using one of the preceding problems.)

Notice, from #3, that $M = PDP^{-1}$ diagonal

So $M^k = \underbrace{PD\cancel{P^{-1}}\,\cancel{P}D\cancel{P^{-1}}\cdots\cancel{P}DP^{-1}}_{k \text{ times}} = PD^kP^{-1}$

So

$$M^{10} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7^{10} & 0 \\ 0 & 5^{10} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \text{something kinda ugly}$$

8. Let $A = \begin{bmatrix} 5 & -3 \\ 0 & 5 \end{bmatrix}$. Find an invertible matrix P and a *diagonal* matrix D such that $A = PDP^{-1}$ or explain why this is impossible.

Char. poly. : $P_A(t) = (5-t)^2$ $\lambda = 5$ only eigenvector

$E_5(A)$:

$$A - 5I = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathcal{B}_{E_5(A)} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Only ONE linearly independent eigenvector, so it's impossible to form an invertible matrix P.

9. (a) **Definition.** An $n \times n$ matrix A is DIAGONALIZABLE if we can write $A = PDP^{-1}$ where D is a diagonal matrix. Go through the last few problems and determine whether the given matrices had this property or not.

(b) **Theorem. (6.9)** Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if there is a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ where each $\vec{v}_i$ is an eigenvector of A.

Explain why this theorem is true. (Hint: Think about how we've been constructing D and especially P in the previous examples.)

D = always possible to construct. Just put eigenvalues along diagonal (put as many as multiplicity of each eigenvalue in the characteristic polynomial).

P = eigenvectors corresponding to the entries of D.

If there are n linearly independent eigenvectors, then P is invertible. Otherwise it's not.

(Unifying Thm)

10. (a) **Theorem A. (6.10)** Let A be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be *distinct* eigenvalues of A and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be corresponding eigenvalues (i.e., $A\vec{v}_i = \lambda_i \vec{v}_i$). Then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is

LINEARLY INDEPENDENT

(b) **Theorem B. (6.11)** Let A be an $n \times n$ matrix. For each eigenvalue λ of A, say that m_λ is the multiplicity of λ in $P_A(t)$. Then A is diagonalizable if and only $\dim(E_\lambda(A)) = m_\lambda$ for each eigenvalue λ.

Assume that Theorem A is true. Explain why Theorem B is true.

This is really just an extension of the theorem in #9.

Theorem A says that eigenvectors corresponding to distinct eigenvalues are linearly independent. This means that to find our vectors for the columns of P, we just have to check that there are enough linearly independent vectors in each $E_\lambda(A)$. In particular, this means that $\dim(E_\lambda(A)) = m_\lambda$ for all λ.

11. Let $P_M(t) = (-2-t)(5-t)(8-t)$.

(a) M is a 3 $\times$ 3 matrix.

(b) The eigenvalues of M are -2, 5, 8.

(c) Is M guaranteed to be diagonalizable? Explain. (Note: See Theorem 6.12 in our textbook.)

IMPORTANT FACT If an nxn matrix has n distinct eigen values, then Theorem B automatically says that the matrix is diagonalizable. So $\boxed{\text{YEP}}$

(d) What is $\det(M)$? Is M invertible?

Notice:

$$\det(M) = \det(M - 0I) = P_M(0) \quad (\text{char. poly evaluated at } t=0)$$

$$= (-2)(5)(8) \neq 0 \quad \text{So invertible.}$$