Week 9 Worksheet

1. (a) Find $P_A(t)$ for $A = \begin{bmatrix} 11 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ -16 & 5 & 2 & 0 & 0 \\ 7 & 19 & 13 & 0 & 0 \\ 6 & 82 & -1 & 1 & 3 \end{bmatrix}$ and list the eigenvalues for A.

   (b) The matrix A is known as a ________________________ matrix. Explain how to find $P_A(t)$ and the eigenvalues for these matrices.

2. (a) Let $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Calculate D^3.

   (b) The matrix D is known as a ________________________ matrix. Explain how to calculate D^k for these matrices.

   (c) Multiplying matrices like D is __________________ when compared to multiplying matrices in general.

3. The matrices P, P^{-1}, and D are given below.

$$P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}$$

(a) Define $A = PDP^{-1}$. Calculate A.

(b) Calculate $P_A(t)$.

(c) Find bases for each of the eigenspaces of A.

4. What conclusions can you make in the last problem? What are the entries in D in relation to A? What are the columns of P in relation to A? Is there any relationship between the entries of D and the columns of P?

Goal: Given a matrix A, find P (invertible) and D (diagonal) where $A = PDP^{-1}$. To try this right away, skip the next problem for now and go to the next page.

5. **The Unifying Theorem: The Final Chapter**

   Let A be an $n \times n$ matrix. A few of the many parts of the Unifying Theorem are:

$$A \text{ is invertible} \iff \text{null}(A) = \left\{\vec{0}\right\} \iff \det(A) \neq 0$$

   With this in mind, try to add something about eigenvalues to the Unifying Theorem. *Hint:* Either consider $E_\lambda(A)$ or $P_A(t)$, and come up with a string of a few "if and only if" statements. You should start with one of the statements above (or its negation) and end with a statement about eigenvalues.

6. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 3 \\ 3 & 0 & 5 \end{bmatrix}$. We are told that $P_A(t) = (2-t)^2(5-t)$. Find an invertible matrix P and a *diagonal* matrix D such that $A = PDP^{-1}$. (Hint: Reverse engineer the process in Problem 3.)

7. Let $M = \begin{bmatrix} 11 & -12 \\ 2 & 1 \end{bmatrix}$. What is M^{10}? (You should be able to answer this question relatively quickly using one of the preceding problems.)

8. Let $A = \begin{bmatrix} 5 & -3 \\ 0 & 5 \end{bmatrix}$. Find an invertible matrix P and a *diagonal* matrix D such that $A = PDP^{-1}$ or explain why this is impossible.

9. (a) **Definition.** An $n \times n$ matrix A is ________________________ if we can write $A = PDP^{-1}$ where D is a diagonal matrix. Go through the last few problems and determine whether the given matrices had this property or not.

   (b) **Theorem. (6.9)** Let A be an $n \times n$ matrix. Then A is ________________________ if and only if there is a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ where each $\vec{v}_i$ is an eigenvector of A.

   Explain why this theorem is true. (Hint: Think about how we've been constructing D and especially P in the previous examples.)

10. (a) **Theorem A. (6.10)** Let A be an $n \times n$ matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be *distinct* eigenvalues of A and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be corresponding eigenvalues (i.e., $A\vec{v}_i = \lambda_i \vec{v}_i$). Then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is

______________________________.

(b) **Theorem B. (6.11)** Let A be an $n \times n$ matrix. For each eigenvalue λ of A, say that m_λ is the

multiplicity of λ in $P_A(t)$. Then A is diagonalizable if and only __________________ $= m_\lambda$ for each eigenvalue λ.

Assume that Theorem A is true. Explain why Theorem B is true.

11. Let $P_M(t) = (-2 - t)(5 - t)(8 - t)$.

(a) M is a ______ $\times$ ______ matrix.

(b) The eigenvalues of M are ______________________________.

(c) Is M guaranteed to be diagonalizable? Explain. (Note: See Theorem 6.12 in our textbook.)

(d) What is $\det(M)$? Is M invertible?