Week 6 Worksheet

1. (a) Let $S = \text{null}\left(\begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}\right)$.

   (i) S __is__ a subspace of __$\mathbb{R}^4$__ because __it's the null space of a matrix__.

   (ii) Rewrite S as the span of one or more vectors.

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 1 & 0 \\ 0 & \underset{\text{free}}{0} & 1 & \underset{\text{free}}{-2} & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_2 = s \\ x_4 = t \end{array} \quad \& \quad \begin{array}{l} x_1 = 3s - t \\ x_3 = 2t \end{array}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 3s-t \\ s \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} t$$

So $\boxed{S = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}}$

(b) Let S be the solution set to

$$\begin{aligned} -2x_1 + 3x_2 &= 0 \\ x_1 - 2x_2 &= 0 \\ 3x_1 + x_2 &= 0 \end{aligned}$$

   (i) S __is__ a subspace of __$\mathbb{R}^2$__ because __it's the solution set to a system of homogeneous equations__

   (ii) Rewrite S as the null space of a matrix.

$$\Leftrightarrow \begin{bmatrix} -2 & 3 \\ 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \quad \text{So} \quad \boxed{S = \text{null}\left(\begin{bmatrix} -2 & 3 \\ 1 & -2 \\ 3 & 1 \end{bmatrix}\right)}$$

(c) Let $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (i) S __is__ a subspace of __$\mathbb{R}^3$__ because __it's the span of vectors in $\mathbb{R}^3$__.

   (ii) Rewrite S as the solution set of a system of homogeneous linear equations.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S \Leftrightarrow \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & y \\ -1 & 0 & z \end{array}\right] \text{ is } \underline{\underline{\text{consistent}}}$$

echelon form: $\left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 1 & y \\ 0 & 0 & x-2y+z \end{array}\right] \rightarrow$ Consistent iff $\boxed{x - 2y + z = 0}$

only one equation.

2. Let $S = \operatorname{span} V$ where $V = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Just by looking at the *number* of vectors in V, we know that $\dim S$ $\underline{\leq 5}$. Explain:

dim(S) = # vectors in a basis.

Since 5 vectors span S, a basis contains at most 5 vectors

(b) Since $S \subseteq \mathbb{R}^4$, we actually know that $\dim S$ $\underline{\leq 4}$. Explain:

Since $S \subseteq \mathbb{R}^4$, we know that

$$\dim(S) \leq \dim(\mathbb{R}^4) = 4$$

(c) Use "Option 1" from Wednesday's class to find a basis for S.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -2 & 0 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \boxed{\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

is a basis for S

(d) Use "Option 2" from Wednesday's class to find a basis for S.

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 1 & -2 & 0 & 1 & 1 \end{bmatrix} \sim \begin{array}{c} \begin{matrix} * & & * & & \end{matrix} \\ \begin{bmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow \boxed{\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$ is a basis for S

WARNING:
OPTIONS 1 & 2 usually give different bases from each other!

(e) From either (c) or (d), we now know that $\dim S$ $\underline{= 2}$.

3. Let S be a subspace of $\mathbb{R}^n$.

Many possible answers

(a) Why might it be helpful to write S as the span of some vectors?

- From here, "easy" to find a basis for S.
- Easy to "generate" (i.e. come up with) vectors in S.

(b) Why might it be helpful to write S as the null space of a matrix (or the solution to a system of homogeneous linear equations)?

- Easy to check whether a given vector is in S!
- If matrix in echelon form, can determine dimension of S ($\leftarrow$ How???)

4. This question is about subspaces **in general**. However, it might be helpful to consider the example $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ while answering them.

(a) Can using "Option 1" change the vectors in question? Can "Option 2" change the vectors?

- Option 1: Yes, in the given example it produces $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ not originally listed!
- Option 2: No, it only throws out vectors that made the set linearly independent (here $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$)

(b) Why do we use the *original columns* in "Option 2"? Can we use the original rows in "Option 1"?

- Option 2: In this example: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

  BUT $S \neq \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

  ROW OPERATIONS CAN CHANGE SPAN OF COLUMNS!

- Option 1: Must use new rows; in this example the old rows are linearly independent.

5. Define $S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \text{ and } y - z = 0 \right\}$ and $S_2 = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

(a) One of these subspaces is contained in the other. Determine which and explain why.

It helps to have them written in the same way:

$\underline{\underline{S_2}}$: $\left[\begin{array}{cc|c} -1 & 1 & x \\ -1 & 2 & y \\ 1 & -1 & z \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & -x \\ 0 & 1 & -x+y \\ 0 & 0 & x+z \end{array}\right] \Rightarrow \boxed{S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x+z=0 \right\}}$

Notice that if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S_1$ then $\begin{cases} x-y+2z=0 \\ y-z=0 \end{cases}$ so (adding these together), then $x+z=0$. This means that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S_2$

So $\boxed{S_1 \subseteq S_2}$!

(b) Find a basis for the subspace that is contained in the other subspace.

Finding basis for S_1:

$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow$ Free: $z = t$ and then $x = -t$, $y = t$

$\Rightarrow S_1 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ since this one vector is linearly independent $\boxed{\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}}$

(c) Extend the basis that you found in Part (b) to a basis for the other subspace.

Use OPTION 2! Notice that $S_2 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$

why could I add this? Does it change Span??

so:

$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \boxed{\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}}$

See Last Page for more on this problem

6. For each of the following, $\mathcal{B}$ is not a basis for S. For each, explain why.

(a) $S = \text{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$ $\quad \mathcal{B} = \text{span}\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\}$.

Here $\mathcal{B}$ is the span of some vectors, so there are infinitely many vectors in $\mathcal{B}$ and thus $\mathcal{B}$ cannot be linearly independent.

(b) $S = \{\vec{x} \in \mathbb{R}^4 \ : \ x_1 + x_3 - x_4 = 0\}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \right\}$.

$S = \text{span}\left\{ \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \right\}$ ← these 3 are linearly independent

So $\dim S = 3$

Since $\mathcal{B}$ only contains 2 vectors, it can't span S.

(c) $S = \mathbb{R}^2$ $\quad \mathcal{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix} \right\}$.

These vectors just don't exist in $\mathbb{R}^2$, so it's not happening. (i.e., $\mathcal{B} \not\subseteq S$ which is bad news)

(d) $S = \text{null}\begin{bmatrix}1 & 0 & 2 & 1\\0 & 1 & -1 & 0\end{bmatrix}$ $\quad \mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\1\\1\\-2\end{bmatrix}} \right\}$.

$\mathcal{B}$ is actually linearly dependent because $\vec{v}_1 - 2\vec{v}_2 - \vec{v}_3 = \vec{0}$

So $\mathcal{B}$ can't be a basis.

(BTW, here $\dim(S) = 2$)

7. **The Unifying Theorem**

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of vectors in $\mathbb{R}^n$. Define $A = [\vec{a}_1 \dots \vec{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined $T(\vec{x}) = A\vec{x}$. The following are equivalent.

(a) S spans $\mathbb{R}^n$.
(b) S is linearly independent.
(c) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.
(d) T is onto.
(e) T is one-to-one.
(f) A is invertible.
(f.5) T is invertible.

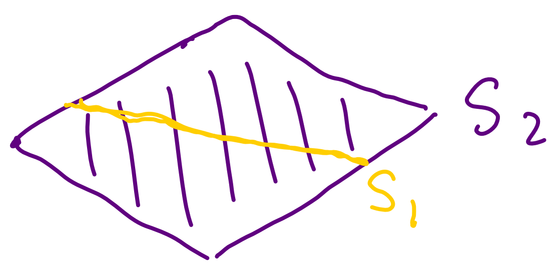
(g) $\ker(T) = \{\vec{0}\}$ (Something involving the kernel of T)

(g.5) $\text{null}(A) = \{\vec{0}\}$ (Something involving the null space of A)

(h) S is a basis for $\mathbb{R}^n$ (Something involving a basis)

More Stuff About #5:

Geometric Picture: Since $\dim S_1 = 1$, $\dim S_2 = 2$, and $S_1 \subseteq S_2$, geometrically S_1 is a line that lies within S_2, which is a plane. Bad picture:



Note on (a):

I could've instead written $S_1 = \text{Span}\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ (see part (b)) and then noticed that $\begin{bmatrix}-1\\1\\1\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}\right\} = S_2$

(Needs a bit of calculations)

This would also imply that $\boxed{S_1 \subseteq S_2}$