Week 6 Worksheet

1. (a) Let $S = \text{null}\left(\begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}\right)$.

   (i) S ______ a subspace of ___ because ____.

   (ii) Rewrite S as the span of one or more vectors.

   (b) Let S be the solution set to

$$\begin{aligned} -2x_1 + 3x_2 &= 0 \\ x_1 - 2x_2 &= 0 \\ 3x_1 + x_2 &= 0 \end{aligned}$$

   (i) S ______ a subspace of ___ because ____.

   (ii) Rewrite S as the null space of a matrix.

   (c) Let $S = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$.

   (i) S ______ a subspace of ___ because ____.

   (ii) Rewrite S as the solution set of a system of homogeneous linear equations.

2. Let $S = \operatorname{span} V$ where $V = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Just by looking at the *number* of vectors in V, we know that $\dim S$ __________. Explain:

(b) Since $S \subseteq \mathbb{R}$—, we actually know that $\dim S$ __________. Explain:

(c) Use "Option 1" from Wednesday's class to find a basis for S.

(d) Use "Option 2" from Wednesday's class to find a basis for S.

(e) From either (c) or (d), we now know that $\dim S$ __________.

3. Let S be a subspace of $\mathbb{R}^n$.
   (a) Why might it be helpful to write S as the span of some vectors?

   (b) Why might it be helpful to write S as the null space of a matrix (or the solution to a system of homogeneous linear equations)?

4. This question is about subspaces **in general**. However, it might be helpful to consider the example $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ while answering them.
   (a) Can using "Option 1" change the vectors in question? Can "Option 2" change the vectors?

   (b) Why do we use the *original columns* in "Option 2"? Can we use the original rows in "Option 1"?

5. Define $S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \text{ and } y - z = 0 \right\}$ and $S_2 = \operatorname{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

(a) One of these subspaces is contained in the other. Determine which and explain why.

(b) Find a basis for the subspace that is contained in the other subspace.

(c) Extend the basis that you found in Part (b) to a basis for the other subspace.

6. For each of the following, $\mathcal{B}$ is not a basis for S. For each, explain why.

(a) $S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ $\quad \mathcal{B} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(b) $S = \{\vec{x} \in \mathbb{R}^4 \ : \ x_1 + x_3 - x_4 = 0\}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) $S = \mathbb{R}^2$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

(d) $S = \operatorname{null} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$ $\quad \mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}$.

7. **The Unifying Theorem**

Let $S = \{\vec{a}_1, \ldots, \vec{a}_n\}$ be a set of vectors in $\mathbb{R}^n$. Define $A = [\vec{a}_1 \ldots \vec{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined $T(\vec{x}) = A\vec{x}$. The following are equivalent.

(a) S spans $\mathbb{R}^n$.
(b) S is linearly independent.
(c) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.
(d) T is onto.
(e) T is one-to-one.
(f) A is invertible.
(f.5) T is invertible.

(g) ______________________________ (Something involving the kernel of T)

(g.5) ______________________________ (Something involving the null space of A)

(h) ______________________________ (Something involving a basis)