Week 1 Worksheet

0. Do the mini quiz problem. Put your work and solution here:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 2 & -1 & 4 \\ -1 & 1 & 4 & 2 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 2 \\ -1 & 1 & 4 & 2 \end{array}\right] \underset{\sim}{R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \end{array}\right]$$

$$\underset{\sim}{-3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

leading (pivot) free

$x_3 = t$ & solve from bottom to top

$x_2 = 2 - x_3 = 2 - t$

$x_1 = 4 - 2x_2 + x_3$
$= 4 - 2(2-t) + t = 3t$

Solution: $(3t, 2-t, t)$ for any $t \in \mathbb{R}$

1. Given a system of equations, is it guaranteed that the Gaussian Elimination process will eventually stop? Explain your answer. Use your answer to make some connection between systems in general and systems in echelon form.

Yes – at step 4 we consider a smaller matrix. So after some number of steps, we will run out of rows & columns, so we'll be done.
This says that EVERY system is equivalent to some system in echelon form.

2. (a) A system is in **reduced echelon form** if
   (i) it's in echelon form,
   (ii) every pivot is 1, and
   (iii) the only nonzero entry in each pivot column is the pivot itself.

   (b) With part (a) in mind, describe how you can modify Gaussian Elimination to find an equivalent system in reduced echelon form. This is called **Gauss-Jordan elimination.**

   STEP 1 Perform Gaussian Elimination to get an equivalent system in echelon form

   STEP 2 Starting at the bottom, use row op. (c) to remove all nonzero entries above each pivot.

3. Use Gauss-Jordan elimination to put the following system into reduced echelon form. Use this to solve the system.

$$\begin{aligned} y + z &= 3 \\ x + y - 2z &= 0 \\ x + 2z &= 2 \end{aligned}$$

Aug. matrix: $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 3 \\ 1 & 1 & -2 & 0 \\ 1 & 0 & 2 & 2 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{array}\right] \overset{-R_1+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 4 & 2 \end{array}\right]$

$\overset{R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 5 \end{array}\right] \overset{\frac{1}{5}R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{-R_3+R_2 \to R_2 \ \& \ 2R_3+R_1 \to R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\overset{-R_2+R_1 \to R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \boxed{\text{Solution is } (x,y,z) = (0,2,1)}$

4. (Number of solutions; Theorem 1.3 in textbook.) **Theorem:** A system of equations has either

(a) NO solutions,

(b) exactly one solution, or

(c) infinitely many solutions

Use tools we have developed (augmented matrices, Gaussian-Elimination, echelon form) to explain why this theorem is true.

Start by putting the system into an augmented matrix & then use G-E to put it into echelon form. There are only two possibilities to consider:

OPTION 1. There is a row $[0\ 0 \dots 0 \mid c]$ where $c \neq 0$ in echelon form. In this case the system has NO SOLUTIONS.

OPTION 2: There are no rows like in option 1. Therefore we can find some solution (ID free variables & back substitute). If there are no free variables, there's EXACTLY ONE SOLUTION. If there's at least one free variable, there's INFINITELY MANY SOLUTIONS!

5. (GeoGebra) Consider Problem 3. Put the given three planes into GeoGebra. Then put *each* new plane that your row operations created into GeoGebra as well. Notice that row operations (a) and (b) do not change the planes, but row operation (c) does. However, what does row operation (c) keep the same? (To help answer this, just look at the two rows used in operation (c) and determine what happens to them.)

Say $a_1x_1+a_2x_2+\dots+a_nx_n=b$ and $c_1x_1+c_2x_2+\dots+c_nx_n=d$ are the two equations. Their solution sets are HYPERPLANES and the solution set of both together is the intersection of these hyperplanes. When we perform row op. (c), we change one of these hyper-planes BUT the intersection stays the same!

6. The points $(1,-1,0)$, $(1,0,2)$, and $(0,-1,1)$ all lie on a plane in $\mathbb{R}^3$. This plane can be written

$$ax+by+cz=d.$$

Use techniques from class to determine what a,b,c,d are. A point is on the plane iff plugging in its coordinates makes the equation true.

$\rightsquigarrow$ We need: * $a-b=d$ $a-b-d=0$

* $a+2c=d$ $\Rightarrow$ $a+2c-d=0$

* $-b+c=d$ $-b+c-d=0$

$$\Rightarrow\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0\\ 1 & 0 & 2 & -1 & 0\\ 0 & -1 & 1 & -1 & 0\end{array}\right] \quad \text{PERFORM GAUSS-JORDAN ELIMINATION} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0\\ 0 & 1 & 0 & 2/3 & 0\\ 0 & 0 & 1 & -1/3 & 0\end{array}\right]$$

Solution: $(a,b,c,d)=\left(t/3,\ -2t/3,\ t/3,\ t\right)$ for any $t\in\mathbb{R}$

Is there more than one possibility for a,b,c,d? Do these different values correspond to different planes?

If $t=0$, then this isn't actually a plane. (Why?) For any other t, it's the same plane. For example, $t=3$ gives us $\boxed{x-2y+z=3}$.