Math 461

Friday, February 14

Chapter 9 In-class problems I

0. (a) Explain the difference between a *trail*, a *walk*, and a *path* in a graph. What does it mean for a trail or walk to be *closed*? Can a path be closed?
   (b) Give the definition of a *subgraph*, and an *induced subgraph*.
   (c) Give an example of G, H_1, and H_2 where H_1 is an induced subgraph of G and H_2 is a subgraph of G that is not induced.

1. (a) Let G be a connected graph. Assume v and w are vertices of G are vertices with odd degree and that every other vertex of G has even degree. Is it guaranteed that that G has an Eulerian trail? Either prove or disprove this.
   (b) Assume G has a closed Eulerian trail. Must G have an even number of edges?

2. A *cycle* in a graph is a closed trail that does not touch any vertex twice, apart from the initial/final vertex. A cycle that contains all vertices of a graph is called a *Hamiltonian cycle*. A path that contains all vertices is a *Hamiltonian path*.
   (a) If G has a Hamiltonian cycle, must it have a Hamiltonian path? What about the other way around?
   (b) If a simple graph has a Eulerian trail, must it have a Hamiltonian cycle? What about the other way around?

3. Let G be a simple graph with n vertices.
   (a) What is the *most* edges that G could have? This graph is called ____________ and is denoted:
   (b) Assume G is connected. How *few* edges can G have? This type of graph also has a name:

4. Let G be a simple graph with 10 vertices and 38 edges. Prove that G contains K_4 as an induced subgraph.

5. Let G be a tree on vertices. This means that ____________. How many edges must G have? Prove this! (Hint: First show that every tree has at least one *leaf* vertex, which means ____________. Then use induction.)

6. Let G be a simple graph G. Its *complement* is the graph $\overline{G}$ that has the same set of vertices as G but has precisely the edges that G does not. Find an example of a graph G such that both G and $\overline{G}$ have a Hamiltonian cycle. How "small" can G be?

7. Prove that any simple graph has two vertices with the same degree.

8. The *(ordered) degree sequence* of a graph G is the list of the degrees of its vertices listed in weakly decreasing order.
   (a) What kind of object is the degree sequence? (We've studied this object already in this class.) Tell me about this object in terms of the number of vertices and edges of G.
   (b) Let λ be the degree sequence of some simple graph. Show that λ is not self-conjugate.

9. Let $n \geq 3$, and let G be a graph on n vertices such that each vertex has degree at least $n/2$.
   (a) Show that G is connected.
   (b) Show that G has a Hamiltonian cycle.