Math 461

Friday, January 31

Chapter 5 In-class problems II

0. Let $n \in \mathbb{N}$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{Z}^\ell$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$, then we say that λ is a(n) ____________________________________.

   This is often denoted $\lambda \vdash n$. We say that λ has ℓ ____________.

   (a) Explain the difference between λ and a composition of n.

   (b) Explain the difference between λ and a set partition of $[n]$.

   (c) I think the book uses $p(n)$ and $p_\ell(n)$ to count two quantities related to this problem. What are these quantities?

1. Compute $p(6)$ by just writing down all of the Ferrers shapes (i.e., Ferrers diagrams, Young diagrams, etc.) for all possible $\lambda \vdash 6$.

2. (a) Give an example of a self-conjugate partition λ of 9. Give a different partition of 9 that is not self-conjugate.

   (b) Prove that the number of partitions of n into exactly ℓ parts is equal to the number of partitions of n with largest part exactly equal to ℓ.

3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$. A *Standard Young Tableau* (SYT) of shape λ is a Ferrers shape of λ that is filled with the numbers 1 through n such that the rows increase from left to right and the columns increase from top to bottom. For example, the left image below is a Standard Young Tableau for $\lambda = (4, 2, 1)$, but the right one isn't (for two separate reasons).

1	3	6	7
2	5		
4			

1	5	7	6
2	4		
3			

   (a) Find all SYT of shape $\lambda = (3, 2)$.

   (b) How many SYT are there of shape $(1, \ldots, 1)$? Of shape (n)?

   (c) Find a formula for the total number of SYT of the partition $\lambda = (m, 1, \ldots, 1)$ (assume there are k 1s).

4. * Assume we have n balls and m boxes. Assuming each box must contain at least one ball, how many ways are there to distribute the n balls into the m boxes if

   (a) both are indistinguishable?

   (b) the balls are indistinguishable but the boxes are distinguishable?

   (c) the balls are distinguishable but the boxes are indistinguishable?

   (d) both are distinguishable?

   For some, we have a nice formula; for others we just have a name (and not a nice closed formula).

5. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$, its *Durfee square* is the largest square of boxes that can fit into the Ferrers shape of λ. For example, the the Durfee square of $(4, 3, 2)$ a 2×2 square, and the Durfee square of $(4, 1, 1)$ is a 1×1 square.

   Given $(\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$, how can we determine the size of its Durfee square without drawing its Ferrers shape?

6. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$. If $p \geq \lambda_1$ and $q \geq \ell$, then we can define the *complement* of λ as follows: Put the Ferrers shape of λ into a $p \times q$ grid of boxes. Consider the boxes *not* in λ and rotate them 180 degrees.

   For example, if $\lambda = (3, 1)$, $p = 3$, and $q = 3$, then the complement of λ is the partition $(3, 2)$.

   (a) Will the complement always be (the Ferrers shape) of a partition? Why?

   (b) What integer is being partitioned by the complement?

   (c) What conditions on p and q guarantee that the complement has the same number of parts as the original?

   (d) What conditions on p and q guarantee that the complement has the same largest part as the original?

   (e) What happens if we complement the complement of λ (using the same p and q)?

7. Show that the number of partitions of n into parts of size at most m is equal to the number of partitions of $n + m$ into m parts.

8. Show that $p_\ell(n + \ell) = p_0(n) + p_1(n) + \cdots + p_\ell(n)$.