Math 461

Wednesday, January 22

Chapter 4 In-class problems I

0. Why are binomial coefficients called binomial coefficients?

1. How many ways are there to place m *indistinguishable* balls into ℓ *distinguishable* boxes?

2. It turns out that $\displaystyle\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$ for any integer $n \geq 2$.

   (a) Prove this fact using the binomial theorem and *calculus*.

   (b) Prove this fact using a "combinatorial proof." That is, find some quantity that is counted by each side of the equation.

   (c) The reading showed that $\displaystyle\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for any integer $n \geq 1$, and the above equation is a generalization of this. Can you generalize this even further? How far? (You don't need to prove it, but you should know how to.)

3. Let $k < n$ be positive integers. Show that $\displaystyle\sum_{i=k}^{n} \binom{n}{i}\binom{i}{k} = 2^{n-k}\binom{n}{k}$ using a combinatorial proof.

4. How many subsets of $[n]$ are strictly larger than their complements?

5. Let $k, m, n \in \mathbb{Z}_{\geq 0}$ such that $k + m \leq n$. Give a combinatorial proof to show that

$$\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m}.$$

6. Fermat's Little Theorem says that if $a \in \mathbb{Z}$ and p is prime, then $a^p \equiv a \pmod{p}$. We'll prove a simplified version of this. (Btw, when I was an undergrad I was convinced that Fermat was a fraud. Look up what he's said about margins in paticular.)

   (a) Let N be a necklace with p beads. We have a colors to use where $a > 1$. How many ways can we color the beads of N using at least two colors? Assume here that any rotation or flip of the necklace is a different coloring.

   (b) Now assume that colorings are the same if one can be obtained from the other via rotation. How many of these are there? How does this prove the above result?

7. A sequence $a_0, a_1, a_2, \ldots$ is **unimodal** if there exists some $i \in \mathbb{Z}_{\geq 0}$ such that $a_j \leq a_{j+1}$ if $j \leq i$ and $a_j \geq a_{j+1}$ if $j > i$. Prove that the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ form a unimodal sequence.

8. Let $k, m, n \in \mathbb{N}$. Show that $\displaystyle\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$.