Daily problems
Assigned 8/14/19
Due 8/16/19

You must attempt each problem, ideally solving as many as possible. We will spend the first part of the next lecture discussing these problems, and you will regularly be asked to present your ideas. (You do not need to fully solve a problem to present it.)

In other words, even if you cannot completely solve a problem, you should work through some related examples or solve a simpler version of the problem for each of these problems, and you should be ready to present your ideas.

(1) Let P be a d-dimensional polytope. Recall that

$$f_i = \# \text{ of } i\text{-dimensional faces of } P.$$

The f**-vector** of P is $f(P) = (f_0, f_1, f_2, \ldots, f_{d-1})$. For each of the following vectors $\mathbf{v}$, either find a polytope P such that $f(P) = \mathbf{v}$ **or** explain why this is impossible.

(a) $(4, 6, 4)$ (So P should have 4 vertices, 6 edges, and 4 "faces", etc.)
(b) $(4, 6, 3)$
(c) $(4, 8, 6)$
(d) $(1, 1, 1, 1)$

(2) Let C_d be the d-dimensional cross-polytope. Give a description of the vertices of C_d, and for each vertex v give a hyperplane H such that $C_d \cap H = \{v\}$.

(3) (a) In this problem, we want to count the number of distinct paths from the point $(0, 0)$ to the point (n, n) subject to the following rules:

- We must stay inside the box $[0, n] \times [0, n]$.
- We move one step at a time, either moving one unit East or one unit North.
- We cannot visit the same point twice.

Example: For $n = 2$, we have the following 6 options:

How many of these paths are there? Try to answer this for $n = 1, 2, 3, 4$, and see if you can find a pattern. (Hint: Look up "binomial coefficients" and think how they might be relevant.)

(b) We want to now count paths that meet the same criteria as Part (a), but with the additional condition:

- The path must **always** stay at or below the line $y = x$.

Notice that in the above example only the first two paths stay at or below the line $y = x$. The others stray above this line at some point.
How many of these paths are there? Try to answer this for $n = 1, 2, 3, 4$. (There is a pattern here, but it's harder to see.)

(c) What does this problem have to do with our class?