Resolving Stanley's conjecture on k-fold acyclic complexes

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Preliminaries: Simplicial Complexes

A **simplicial complex** on n vertices is a subset Δ of $2^{[n]}$ such that

$$\sigma \in \Delta, \tau \subseteq \sigma \implies \tau \in \Delta.$$

f-polynomial:

$$\begin{aligned} f(\Delta, t) &= \sum_{\sigma \in \Delta} t^{|\sigma|} \\ &= f_{-1} + f_0 t + f_1 t^2 + \cdots + f_d t^{d+1} \end{aligned}$$

where f_i is the number of faces of Δ of dimension i.

Preliminaries: Simplicial Complexes

Given complexes Γ and Δ, their **join** is

$$\Gamma \star \Delta = \{\tau \cup \sigma : \tau \in \Gamma \text{ and } \sigma \in \Delta\}$$

If Γ is a $(k-1)$-simplex, then $\Gamma \star \Delta$ is a ***k*-fold cone**. ($k = 1$ is simply a **cone**)

The f-polynomial of a join factors:

$$f(\Gamma \star \Delta, t) = f(\Gamma, t) f(\Delta, t)$$

Preliminaries: Simplicial Homology

$\tilde{H}_i(\Delta, \mathbb{k})$ is the i^{th} **reduced simplicial homology group** of Δ with coefficients in $\mathbb{k}$.

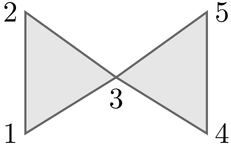
$\tilde{\beta}_i = \dim_{\mathbb{k}} \tilde{H}_i(\Delta, \mathbb{k})$ are the **reduced Betti numbers**. These "count i-dimensional holes" in Δ.

Δ is **acyclic** (over $\mathbb{k}$) if $\tilde{\beta}_i = 0$ for all i.

Acyclicity is topological (up to choice of $\mathbb{k}$).

Preliminaries: An example

$\Delta = \langle 123, 345 \rangle$



$$f(\Delta, t) = 1 + 5t + 6t^2 + 2t^3 = (1 + t)(1 + 4t + 2t^2)$$

Notice that $\Delta = \langle 3 \rangle \star \langle 12, 45 \rangle$, so Δ is a cone. The above factorization is not surprising.

Known results

Theorem (Kalai, 1985)

If Δ is acyclic over some field, then

$$f(\Delta, t) = (1 + t) f(\Delta', t)$$

for some complex Δ'.

$$\{f\text{-vectors of acyclic complexes}\} = \{f\text{-vectors of cones}\}$$

But what is Δ'?

Known results

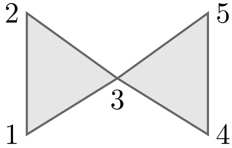
Theorem (Stanley, 1993)

If Δ is acyclic over some field, then Δ can be written as the disjoint union of rank 1 boolean intervals whose minimal faces together form a subcomplex Δ'.

This Δ' is an explicit combinatorial witness to the Δ' that appears in Kalai's result.

Preliminaries: An example

$\Delta = \langle 123, 345 \rangle$

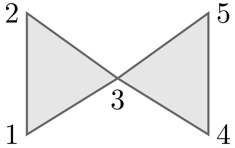


Face poset of Δ:

$$
\begin{array}{ccccccccccc}
 & & & 123 & & & & 345 & & & \\
12 & & 13 & & 23 & & 34 & & 35 & & 45 \\
 & 1 & & 2 & & 3 & & 4 & & 5 & \\
 & & & & & \varnothing & & & & &
\end{array}
$$

Preliminaries: An example

$$\Delta = \langle 123, 345 \rangle$$



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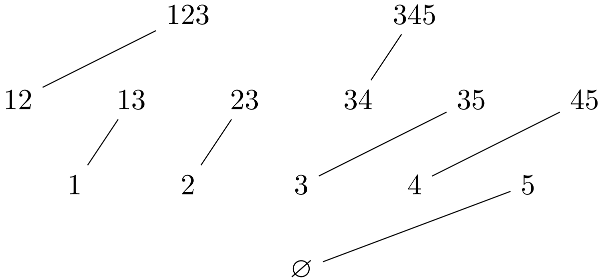

Preliminaries: An example

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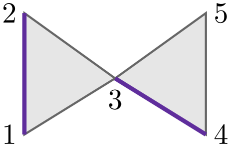
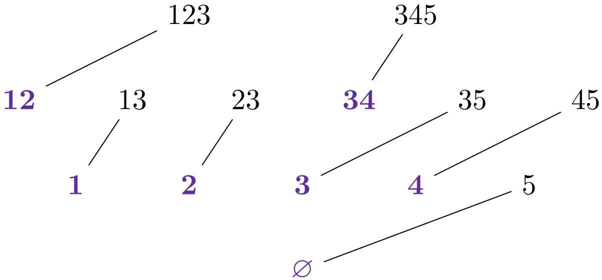


Face poset of Δ:



One last definition

Link of σ: $\operatorname{link}\sigma = \{\tau \in \Delta : \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \varnothing\}$

A complex Δ is ***k*-fold acyclic** if $\operatorname{link}\sigma$ is acyclic for all $\sigma \in \Delta$ such that $|\sigma| < k$.

Acyclicity is equivalent to 1-fold acyclicity. For $k > 1$, this is not topological:

The conjecture

Theorem (Stanley, 1993, follows from Kalai 2001)

If Δ is k-fold acyclic over some field, then $f(\Delta, t) = (1+t)^k f(\Delta', t)$ for some complex Δ'.

$$\{f\text{-vectors of } k\text{-fold acyclic complexes}\} = \{f\text{-vectors of } k\text{-fold cones}\}$$

Conjecture (Stanley, 1993)

*If Δ is **k-fold** acyclic over some field, then Δ can be written as the disjoint union of rank k boolean intervals whose minimal faces together form a subcomplex Δ'.*

Main results

Theorem (Duval, Klivans, and Martin, unpublished)

The conjecture is true for $\dim \Delta \leq 2$.

Theorem (Doolittle and Goeckner, 2018)

The conjecture is false in general.

Remarks:

- We construct an explicit counterexample for $k = 2$ and $\dim \Delta = 3$.
- The conjecture holds for $k = \dim \Delta$. (“Stacked” complexes)
- A slight modification to the statement makes the conjecture true. (Replace “boolean intervals” with “boolean trees”)

Main results

Theorem (Doolittle and Goeckner, 2018)

Let $\Gamma \subseteq \Delta$ *be complexes such that*

1. *Both* Δ *and* Γ *are* k*-fold acyclic,*
2. Γ *is an* ***induced*** *subcomplex, and*
3. *The* ***relative complex*** (Δ, Γ) *cannot be decomposed into rank* k *boolean intervals.*

Then gluing many copies of Δ *together along* Γ *produces a* k*-fold acyclic complex that cannot be decomposed into rank* k *boolean intervals.*

- (1) and (2) preserve simplicialness and k-fold acyclicity; (3) forces the resulting complex to not be decomposable into rank k boolean intervals.
- "Many" > (total number of faces of Γ)$/2^k$

Not the counterexample

$$\Sigma = \langle 1234, 1235, 2345, 2456, 3456 \rangle$$
$$\Upsilon = \langle 125, 124, 246, 346 \rangle$$
$$\Psi = (\Sigma, \Upsilon)$$

- Σ is a triangulation of the octahedron with no interior vertices.
- Υ is a path of triangles on the boundary of Δ.
- Both Σ and Υ are 2-fold acyclic.
- (Σ, Υ) cannot be decomposed into rank 2 boolean intervals.

Not the counterexample

$$\Sigma = \langle 1234, 1235, 2345, 2456, 3456 \rangle$$
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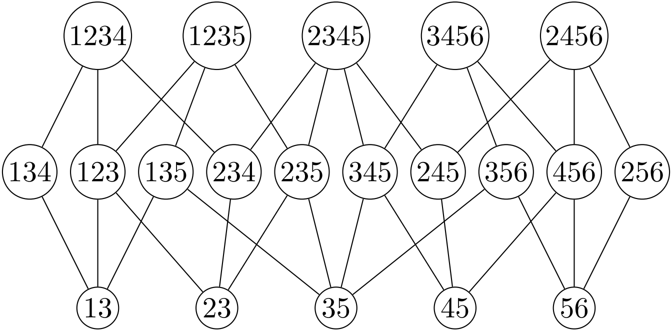

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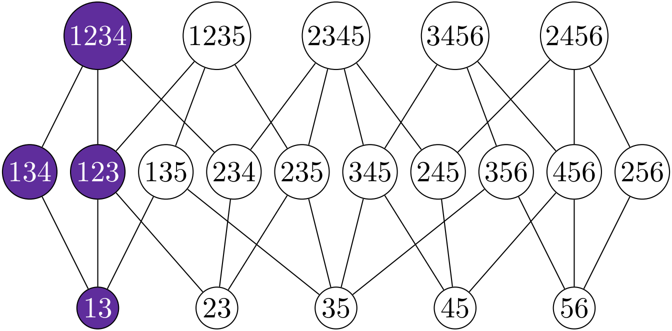

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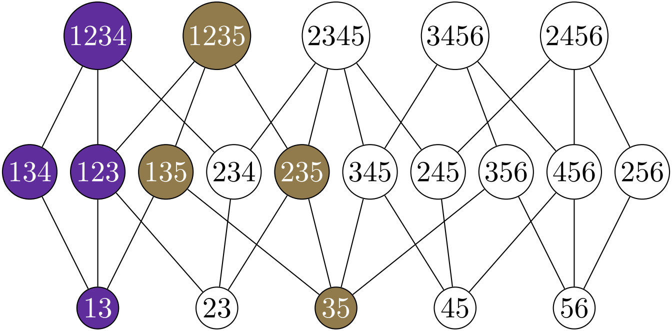

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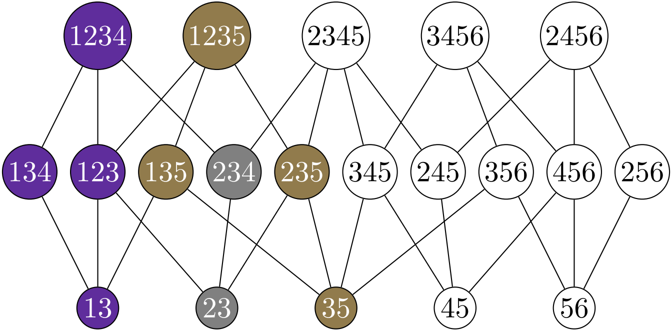


Only problem: Γ is not induced

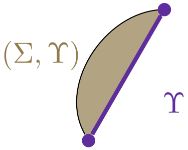
Building the counterexample

$$\Sigma = \langle 1234, 1235, 2345, 2456, 3456 \rangle$$

$$\Upsilon = \langle 125, 124, 246, 346 \rangle$$

$$\Psi = (\Sigma, \Upsilon)$$

Schematic:



Building the counterexample

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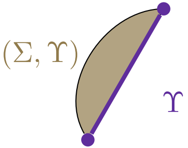

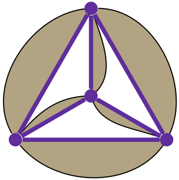

Building the counterexample

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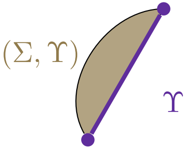
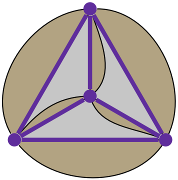


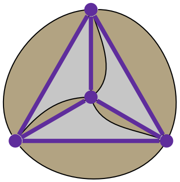


Building the counterexample

Theorem (Doolittle and Goeckner, 2018)

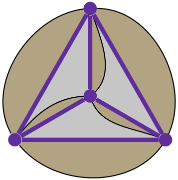
If Δ = *gold* + *purple* + *gray and* Γ = *purple* + *gray, then*

1. *Both* Δ *and* Γ *are 2-fold acyclic,*
2. Γ *is an induced subcomplex, and*
3. *The relative complex* (Δ, Γ) *cannot be decomposed into rank* 2 *boolean intervals.*



Building the counterexample

$\Delta =$ gold $+$ purple $+$ gray and $\Gamma =$ purple $+$ gray



Since Γ has 64 total faces and $64/2^2 = 16$, gluing at least 17 copies of Δ together along Γ will produce a counterexample.

In fact, a linear programs shows that gluing just **three** copies of Δ together along Γ produces a complex that is 2-fold acyclic but not decomposable into rank 2 boolean intervals!

f-polynomial $= 1 + 20t + 136t^2 + 216t^3 + 99t^4 = (1+t)^2(1+18t+99t^2)$

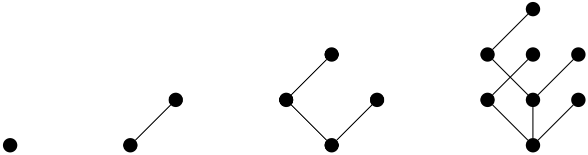
The end

Thanks!

Boolean Trees

A **boolean tree of rank k** is a subposet of a poset P that is defined recursively:

- A rank 0 boolean tree is simply an element of P.
- Given T_1 and T_2, both boolean trees of rank $k-1$ with minimal elements r_1 and r_2 such that r_2 covers r_1, then $T_1 \cup T_2$ is a boolean tree of rank k.



The boolean tree version

Conjecture (Stanley, 1993)

If Δ is k-fold acyclic over some field, then Δ can be written as the disjoint union of rank k boolean intervals whose minimal faces together form a subcomplex Δ'.

Theorem (Doolittle and Goeckner, 2018)

If Δ is k-fold acyclic over some field, then Δ can be written as the disjoint union of rank k boolean ***trees*** *whose minimal faces together form a subcomplex Δ'.*

Proof ideas: Algebraic shifting (Kalai) and iterated homology (Duval–Rose and Duval–Zhang).

The actual end

Thanks again!