Partition extenders, skeleta of simplices, and Simon's conjecture

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Simplicial complexes

Simplicial complex: Collection Δ such that

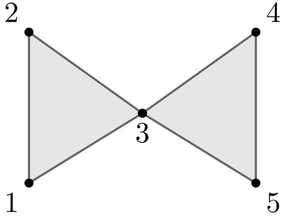
$$\text{if } \sigma \in \Delta \text{ and } \tau \subseteq \sigma, \text{ then } \tau \in \Delta.$$

Face: Element $\sigma \in \Delta$. **Facet**: Maximal element $F \in \Delta$.

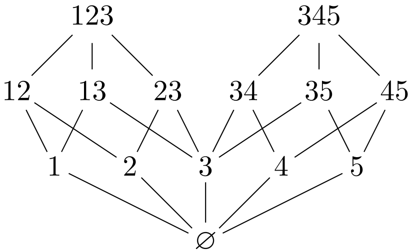
Dimension: $\dim \sigma := |\sigma| - 1$, $\dim \Delta := \max \{\dim \sigma \mid \sigma \in \Delta\}$.

Pure: All facets have the same dimension.

An example



$\Delta = \langle 123, 345 \rangle$



$f(\Delta) = (1, 5, 6, 2)$ **f-vector**: $f_i = \#$ of i-dimensional faces of Δ.

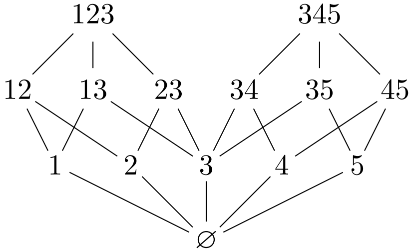
$h(\Delta) = (1, 2, -1, 0)$ **h-vector**: Invertible transformation of f-vector.

Partitionability

Partitionable: Can write Δ as disjoint union of boolean intervals

$$\Delta = [R_1, F_1] \sqcup \cdots \sqcup [R_k, F_k]$$

where $F_1, \ldots, F_k$ are the **facets** of Δ and $[A, B] = \{C \mid A \subseteq C \subseteq B\}$.



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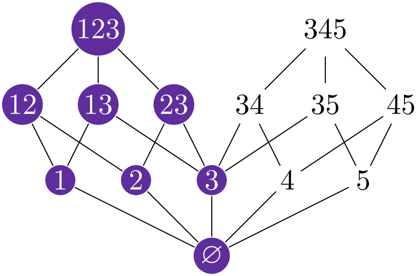

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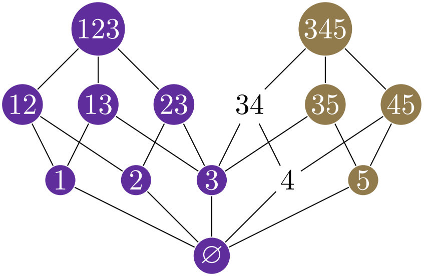

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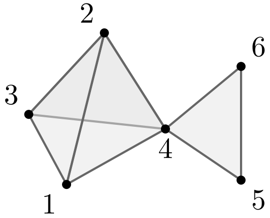
Shellable $\implies$ Partitionable.

Proposition

*If Δ is pure and partitionable, then h_k counts the number of minimal faces R_i of size k in **any** partitioning of Δ.*

The h-vector can also be obtained from the Hilbert series of $\Bbbk[\Delta]$, the **Stanley–Reisner ring** of Δ.

Another example



$$\Delta = \langle 123, 124, 134, 234, 456 \rangle$$

$$f(\Delta) = (1, 6, 9, 5)$$

$$h(\Delta) = (1, 3, 0, 1)$$

This complex is partionable but **not** shellable (or constructible, Cohen–Macaulay, etc.).

$$\Delta = [\varnothing, 456] \sqcup [1, 124] \sqcup [2, 234] \sqcup [3, 134] \sqcup [123, 123]$$

Our question

Proposition

*If Δ is pure and partitionable, then h_k counts the number of minimal faces R_i of size k in **any** partitioning of Δ.*

Goal: Combinatorial interpretation of $h(\Delta)$ when Δ is not partitionable.

Main idea: Relative complexes.

Partition extenders

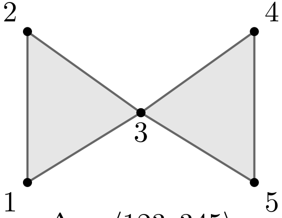
Let $\Delta \subseteq \Gamma$. The **relative complex** (Γ, Δ) is the set of all faces $\sigma \in \Gamma \setminus \Delta$. **Partitionability** is defined as before.

Definition

Let Δ be a pure complex. A ***partition extender*** *for Δ is a pure complex Γ such that*

1. $\Delta \subseteq \Gamma$,
2. $\dim \Gamma = \dim \Delta$, *and*
3. ***both*** *Γ and (Γ, Δ) are partitionable.*

Partition extenders: An example revisited



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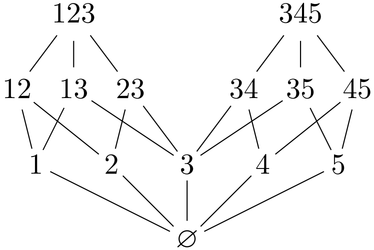

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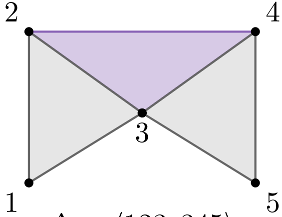
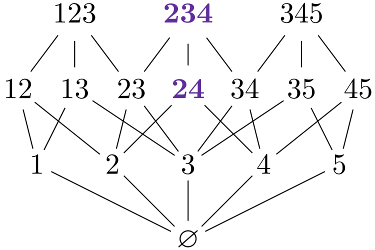
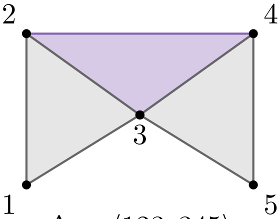


$$\Delta = \langle 123, 345 \rangle$$
$$\Gamma = \langle 123, 345, \mathbf{234} \rangle$$



Partition extenders: An example revisited



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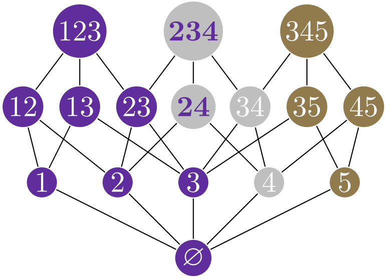

If Γ is a partition extender for Δ, then $h(\Delta) = h(\Gamma) - h(\Gamma, \Delta)$.

Partition extenders

Theorem (Doolittle–G.–Lazar)

Let Δ be a pure complex. Then Δ has a partition extender.

Corollary (Doolittle–G.–Lazar)

The h-vector of any pure complex can "naturally" be written as the difference of two h-vectors of partitionable (relative) complexes.

- Our construction adds **many** faces to construct Γ.
- Is there a **minimal** partition extender? (With respect to added facets, vertices, faces overall?)
- Are minimal partition extenders **unique** in some sense?

Our construction: An example

Fact: A graph is partitionable if and only if it has at most one acyclic component.

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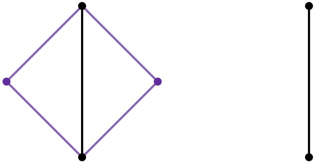

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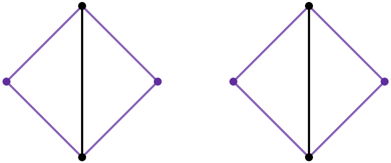

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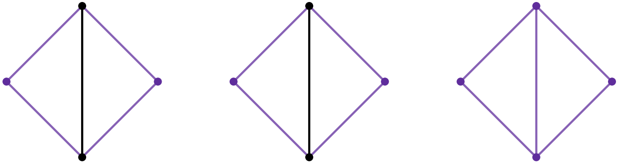

Cohen–Macaulay extenders

Similar notions can be studied for properties that are defined for both simplicial complexes and relative complexes.

Theorem (Doolittle–G.–Lazar)

Let Δ be a pure complex with Stanley–Reisner ring $\Bbbk[\Delta]$. Then Δ has a ***Cohen–Macaulay extender*** *if and only if* $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$.

Depth and the Cohen–Macaulay (CM) property can be defined in terms of (relative) homologies of certain subcomplexes.

If $\operatorname{depth} \Bbbk[\Delta] = \dim \Bbbk[\Delta] - 1$, then **any** CM complex of the same dimension that contains Δ is a CM extender. **In particular**, the skeleton of a simplex works.

Shellable extenders and Simon's conjecture

Conjecture (Doolittle–G.–Lazar)

Let Δ *be a pure complex such that* $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$ *for every field* $\Bbbk$*. Then* Δ *has a* ***shellable extender****.*

Can we always construct shellable extenders without introducing new vertices (as in the CM case)?

If so, this would prove **Simon's conjecture**.

Simon's conjecture

Conjecture (Simon '94)

The d-skeleton of an n-simplex is extendably shellable for all n and d.

Extendably shellable: Any partial shelling can be completed to a full shelling.

- Trivially true for $d \leq 1$ and $d \geq n - 1$.
- True for $d = 2$ and holds for all rank 3 matroids (Björner and Eriksson '94).
- True for $d = n - 2$ (Bigdeli, Yazdan Pour, and Zaare-Nahandi '19 and Dochtermann '21).
- Not all matroids are extendably shellable: The 12-dimensional crosspolytope is not (Hall '04).

Simon's conjecture – Future directions

Relative complexes: Natural setting for overall approach; help in searching for counterexamples.

Lex shellable complexes:

- Weakening of matroid characterization (all vertex orders induce a shelling).
- Implies EL-shellability of face poset.
- Incomparable with vertex decomposability.

The end

Grazie e buona serata!