Manifold matching complexes

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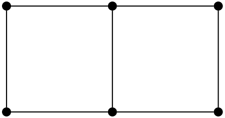
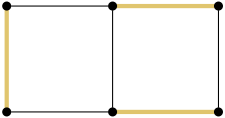
Graphs and matchings

Given a graph G, a **matching** is a collection of edges of G such that no two edges share a common endpoint.



G



(Maximal) matching of G

Our graphs are **simple** (no multiple edges or loops) without **isolated vertices** (not the endpoint of any edges).

Simplicial complexes

A **simplicial complex** is a collection of sets Δ with the following property:

$$\text{If } \sigma \in \Delta \text{ and } \tau \subseteq \sigma, \text{ then } \tau \in \Delta.$$

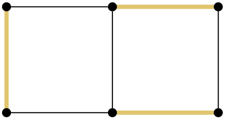
Simplicial complexes are "closed under taking subsets."

Examples/applications:

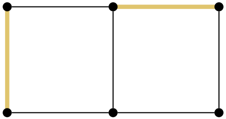
- Linearly independent sets in a vector space
- "Triangulation" of a topological space for computations
- Higher dimensional analogues of simple graphs

Matching complexes

Any subset of a matching is also a matching:



Maximal matching of G

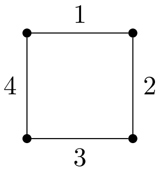


Non-maximal matching of G

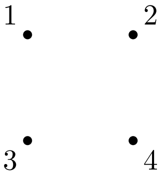
Given a graph G, its **matching complex** is $M(G)$, the simplicial complex of all possible matchings of G.

Example: C_4

The cycle graph C_4 and the **geometric realization** of its matching complex $M(C_4)$.



C_4



$M(C_4)$

Example: C_4

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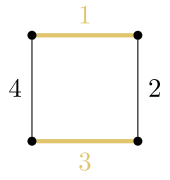
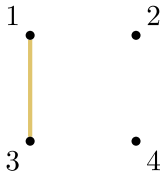


C_4



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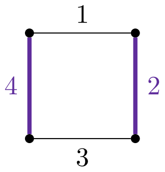
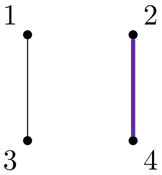


C_4



$M(C_4)$

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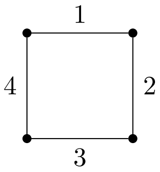
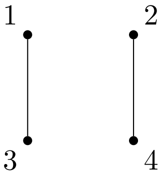


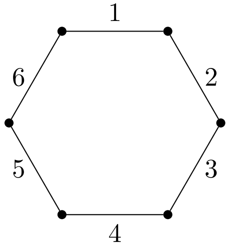
C_4



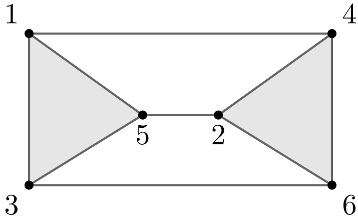
$M(C_4)$

Another one: C_6

The cycle graph C_6 and its matching complex $M(C_6)$.



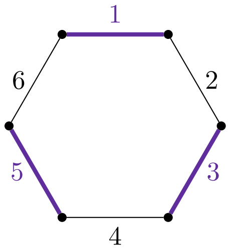
C_6



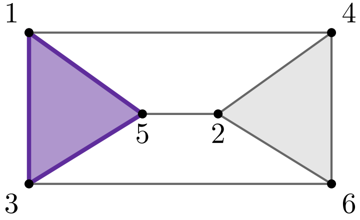
$M(C_6)$

Another one: C_6

The cycle graph C_6 and its matching complex $M(C_6)$.



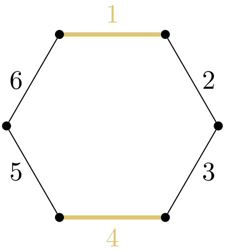
C_6



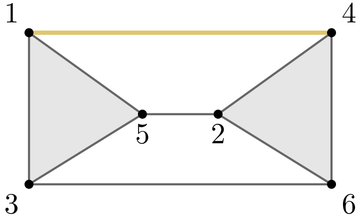
$M(C_6)$

Another one: C_6

The cycle graph C_6 and its matching complex $M(C_6)$.



C_6



$M(C_6)$

The big question

Given G, what is $M(G)$?

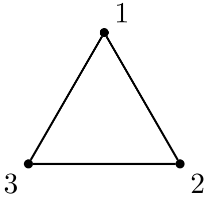
This has been answered for many families of graphs, including **complete graphs** K_n, **complete bipartite graphs** $K_{n,m}$, **cycles** C_n, and **paths** P_n.

The results are **topological** and have surprising and powerful connections to **representation theory**. For example, the homologies of $M(K_n)$ give a formula for irreducible representations of S_n.

The other question

Given $M(G)$, what is G?

More generally, which simplicial complexes are matching complexes?

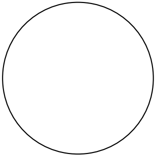


Non-example – matching complexes are **flag**.

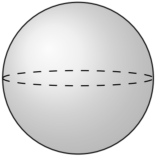
Manifolds

A ***d*-dimensional manifold** is a topological space that "locally looks like $\mathbb{R}^d$."

Examples:



$\mathbb{S}^1$



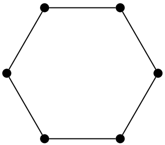
$\mathbb{S}^2$

Spheres (above), planes, torus, Klein bottle, projective plane, etc.

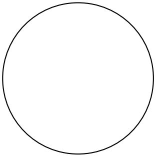
Combinatorial manifolds

A simplicial complex is a **combinatorial manifold** if its geometric realization is homeomorphic to a manifold.*

For example, cycle graphs are homeomorphic to $\mathbb{S}^1$:



C_6

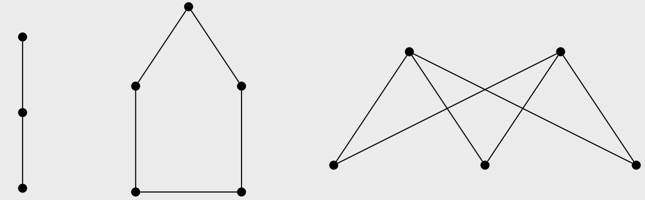


$\mathbb{S}^1$

* - This is a lie, but but true enough for this talk.

Basic sphere graphs

The following graphs are **basic sphere graphs**. All of their matching complexes are homeomorphic to spheres.

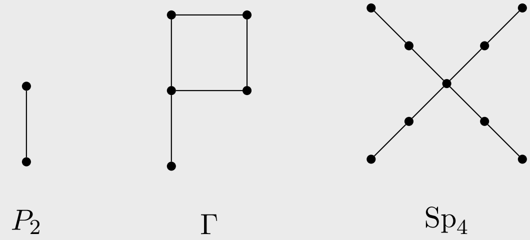


P_3 C_5 $K_{3,2}$

In particular, $M(P_2)$ is two disjoint vertices, $M(C_5) = C_5$, and $M(K_{3,2}) = C_6$.

Basic ball graphs

The following graphs are **basic ball graphs**. All of their matching complexes are homeomorphic to balls (which are **manifolds with boundary**).



The graph Sp_k is a **spider graph** with k legs of length two. Its matching complex is a $(k-1)$-dimensional simplex with k simplices attached to it, which is homeomorphic to a k-dimensional ball.

Our results

Proposition

If G is the disjoint union of basic sphere graphs, then $M(G)$ is a ***sphere****. If G is the disjoint union of basic sphere graphs and at least one basic ball graph, then $M(G)$ is a* ***ball****.*

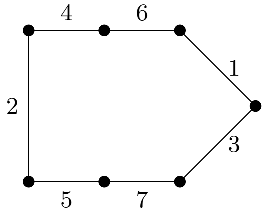
Theorem

If $M(G)$ is a combinatorial manifold and $\dim M(G) \neq 2$, then $M(G)$ is either a sphere or a ball. Moreover, the only graphs G which give such $M(G)$ are as described in the above proposition.

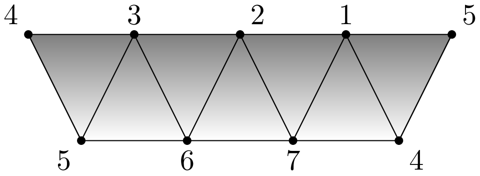
Theorem

In dimension 2 it gets weird.

More examples: $M(C_7)$

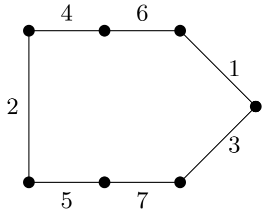


C_7

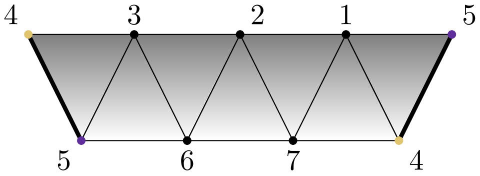


$M(C_7)$

More examples: $M(C_7)$



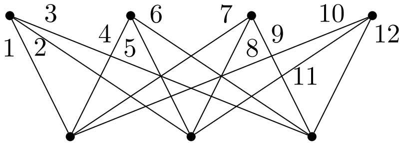
C_7



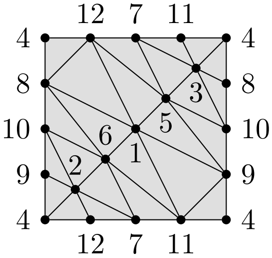
$M(C_7)$

The matching complex of C_7 is a **Möbius strip**! (A manifold with boundary)

More examples: $M(K_{4,3})$

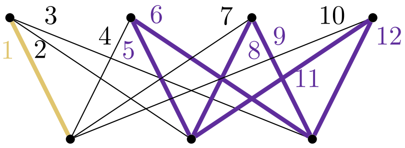


$K_{4,3}$

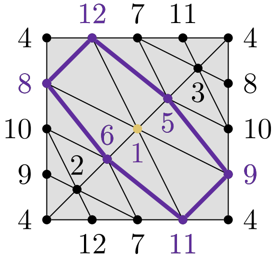


$M(K_{4,3})$

More examples: $M(K_{4,3})$

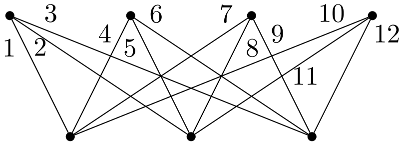


$K_{4,3}$

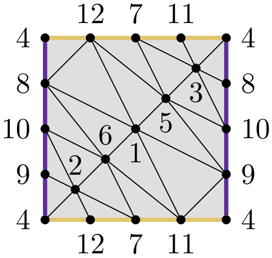


$M(K_{4,3})$

More examples: $M(K_{4,3})$

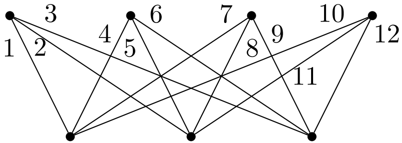


$K_{4,3}$

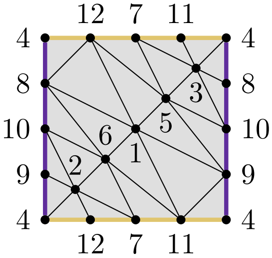


$M(K_{4,3})$

More examples: $M(K_{4,3})$



$K_{4,3}$



$M(K_{4,3})$

$M(K_{4,3})$ is a **torus**! (A manifold without boundary)

Our results in dimension 2

Theorem

Let $M(G)$ be a combinatorial manifold and assume $\dim M(G) = 2$. If $M(G)$ is a ***manifold without boundary****, then either*

1. $M(G) = \mathbb{S}^2$ *and G is a disjoint union of basic sphere graphs, or*
2. $M(G)$ *is a torus and $G = K_{4,3}$.*

If $M(G)$ is a ***manifold with boundary****, then either*

1. $M(G)$ *is a ball and G is a disjoint union of basic ball graphs,*
2. $M(G)$ *is a Möbius strip and G is C_7 or three related graphs,*
3. $M(G)$ *is an annulus and G is two copies of C_4 connected at a vertex, or*
4. $M(G)$ *is a torus with a disk removed and G is one of three graphs.*

Final remarks

Our results were proved by analyzing **links** of faces, which are a specific type of subcomplex. In a combinatorial manifold, the link of every face is a sphere or ball of lower dimension, so we were able to induct on dimension. This leads to several questions:

- Can we classify other matching complexes that generalize manifolds or are defined via link conditions? (Examples: **Buchsbaum** and **vertex decomposable** complexes)
- Matching complexes are **not unique**—for example, C_3 and $K_{3,1}$ have isomorphic matching complexes. Is this the only case where this happens?

Both of these questions are being investigated by Fran Herr and Legrand Jones (University of Washington undergraduates) and Rowan Rowlands (UW graduate student).

The end

Thanks for listening!